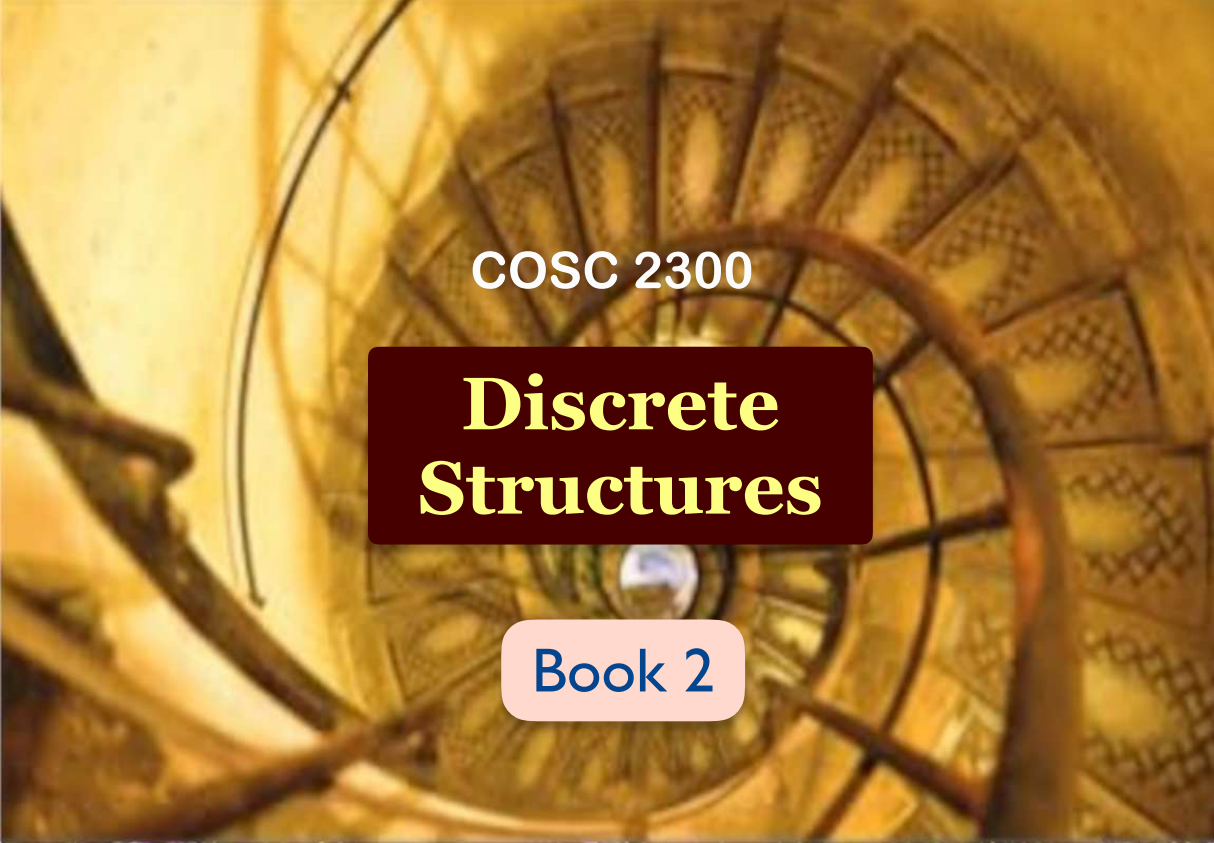




COSC 2300

Discrete Structures

Book 2

If $x, y \in \mathbb{R}$, then the average or mean or arithmetic mean of x and y is $\frac{x+y}{2}$.

Assuming $x, y > 0$, their geometric mean is $\sqrt{xy}$.

eg. the arithmetic mean of 3 and 5 is $\frac{3+5}{2} = 4$; the geometric mean of 3 and 5 is $\sqrt{3 \cdot 5} = \sqrt{15} \approx 3.872983$.

Theorem If $x, y > 0$ then $\underbrace{\sqrt{xy}}_{\text{geometric mean}} \leq \underbrace{\frac{x+y}{2}}_{\text{arithmetic mean}}$. Moreover, equality holds iff $x=y$.

Proof Since $(x-y)^2 \geq 0$ we have $x^2 - 2xy + y^2 \geq 0$ so $x^2 + 2xy + y^2 \geq 4xy$ i.e. $(x+y)^2 \geq 4xy$.
So $x+y \geq \sqrt{4xy} = 2\sqrt{xy}$. Finally, $\frac{x+y}{2} \geq \sqrt{xy}$. $\square$

This is a direct proof... but where did the idea come from?

$$\frac{x+y}{2} \geq \sqrt{xy}$$

$$\Leftrightarrow x+y \geq 2\sqrt{xy}$$

$$\Leftrightarrow (x+y)^2 \geq 4xy$$

$$\Leftrightarrow x^2 + 2xy + y^2 \geq 4xy$$

$$\Leftrightarrow x^2 - 2xy + y^2 \geq 0$$

$$\Leftrightarrow (x-y)^2 \geq 0$$

Does the equation $x^2 + y^2 = 22$ have any integer solution(s)? Justify your answer.

Answer: No. Any integer solution (x, y) can be taken from $x, y \in \{0, 1, 2, 3, 4\}$ without loss of generality. We consider the value of $x^2 + y^2$ for each (x, y) pair in this set:

$$x^2 + y^2 \in \{0, 1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20, 25, 32\}$$

so $x^2 + y^2 \neq 22$.

We used case analysis.

Scratch:

x	0	1	2	3	4
y	0	1	2	3	4
0	0	1	4	9	16
1	1	2	5	10	17
2	4	5	8	13	20
3	9	10	13	18	25
4	16	17	20	25	32

Does the equation $x^2 - y^2 = 22$ have any integer solution(s)? Justify your answer.

Answer: No. We are asking for integer solutions of $(x+y)(x-y) = 22$. Note that

$x+y$ and $x-y$ have the same parity (they are both even or both odd since $(x-y)+2y = x+y$.)

If $x+y$ and $x-y$ are both odd, then $x^2 - y^2 = (x+y)(x-y)$ is odd; but if $x+y$ and $x-y$ are both even, then $(x+y)(x-y)$ is a multiple of 4. However, 22 is neither odd nor a multiple of 4.

Diophantine equations: polynomial equations in several variables for which integer (or rational) solutions are required.

There is no algorithm for answering such problems in general.

If α, β are real irrational, can $\alpha + \beta$ be rational? Justify your answer.

Yes, for example $\underbrace{\pi}_{\text{irrational}} + \underbrace{(4-\pi)}_{\text{irrational}} = 4 \in \mathbb{Q}$.

If you don't know that π is irrational then use this example: $\underbrace{\sqrt{2}}_{\text{irrational}} + \underbrace{(4-\sqrt{2})}_{\text{irrational}} = 4 \in \mathbb{Q}$

If α, β are real irrational, can α^β be rational? Justify your answer.
Yes; for example $\sqrt{2}^{\sqrt{2}} = 2$.

If α, β are real irrational, can α^β be rational? Justify your answer.

If $\sqrt{2}^{\sqrt{2}} \in \mathbb{Q}$ then we're done. Otherwise $\sqrt{2}^{\sqrt{2}} \notin \mathbb{Q}$ and then
 $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2$. This proves that such α, β exist.

Scratch:
Try $\sqrt{2}^{\sqrt{2}}$.

This is a nonconstructive proof of existence of such α, β .

Theorem There exist irrational numbers in $\mathbb{R}$.

Proof Clearly $\mathbb{Q} \subseteq \mathbb{R}$ so $|\mathbb{Q}| \leq |\mathbb{R}|$. If $\mathbb{Q} = \mathbb{R}$ then $|\mathbb{Q}| = |\mathbb{R}|$, contrary to Cantor's theorem that $|\mathbb{Q}| < |\mathbb{R}|$ ($\mathbb{Q}$ is countable but $\mathbb{R}$ is not). $\square$

Proof that e, π are irrational: I have proofs on my website ericmoorhouse.org/handouts

Actually π and e are transcendental. (stronger than irrational).

e^π is irrational. π^e ? Nobody knows.

$e+\pi$, $e\pi$? We know that at least one of $e+\pi$ and $e\pi$ is irrational. This is known because:

$$(x-e)(x-\pi) = x^2 - (e+\pi)x + e\pi$$

e and π are transcendental: not roots of any polynomial in x with rational coefficients.

Algebraic numbers: roots of polynomials with rational coefficients.

$\frac{3}{5}$ is a root of $x - \frac{3}{5}$ so $\frac{3}{5}$ is algebraic.

$\sqrt{2}$ is a root of $x^2 - 2$ so $\sqrt{2}$ is algebraic.

there are only countably many algebraic numbers.

Transcendental numbers exist by the same argument as above using cardinality.

1844 Liouville found a transcendental number.

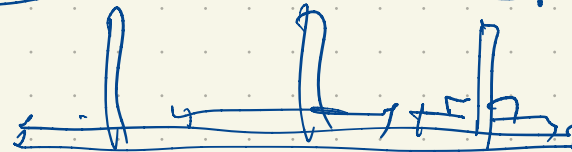
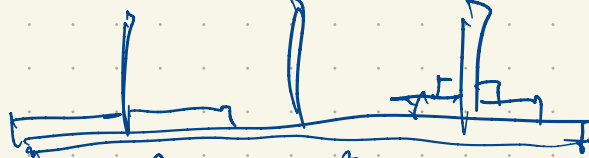
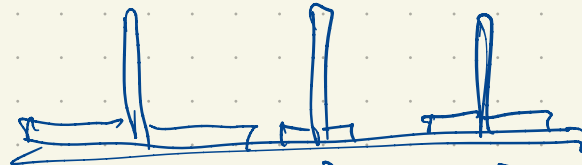
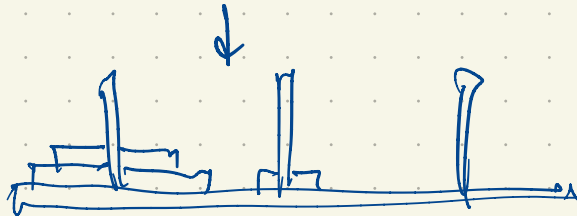
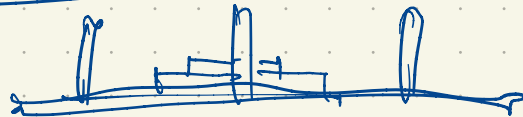
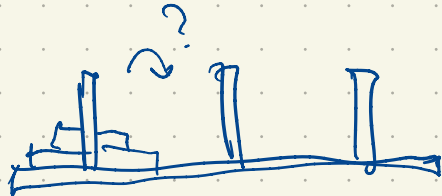
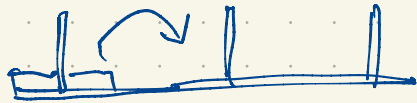
"Is $x^2 \geq 0$ for any real x ?" "Any" and "every" are often interchangeable. Not always.

For every $x \in \mathbb{R}$, $x^2 \geq 0$. } Universal quantifier
For any $x \in \mathbb{R}$, $x^2 \geq 0$. }
For some $x \in \mathbb{R}$, $x^2 = 0$. } Existential quantifier

Is any real number transcendental? Yes, for example e and π .

Is every real number transcendental? No, for example $\frac{3}{5}$, $\sqrt{2}$ are algebraic.

Towers of Hanoi?



Mathematical Induction and Recursion

There is an inductive proof that there exists an algorithm to move a tower of n disks from one stack to another. Beyond this, we can prove (recursive) a formula for the number of steps it takes. (also by induction)