Name Solution Key

Section ☐ **01** Earl-8am ☐ **02** Yeung-9am ☐ **03** Furtado-10am ☐ **04** Li-11am ☐ **05** Furtado-11am ☐ **06** Zhong-12noon ☐ **07** Wiseman-1:10pm ☐ **08** Yeung-2:10pm


# Common Exam I

5:15–7:00pm Thursday February 21, 2019

*Instructions.* Indicate your name and section/instructor above. You may use a scientific non-graphing calculator—no other aids are allowed. Cell phones and other devices must be turned off and left in your backpack/bag during the exam. **Write clearly**, using good mathematical notation and showing all required steps in the space provided. Total value: 100 points.

1. *(10 points)* A graph of the function $f(x) = x\sin\left(\frac{1}{x}\right)$ is shown below.

   Indicate whether each of the following statements is true or false, by circling T or F respectively.

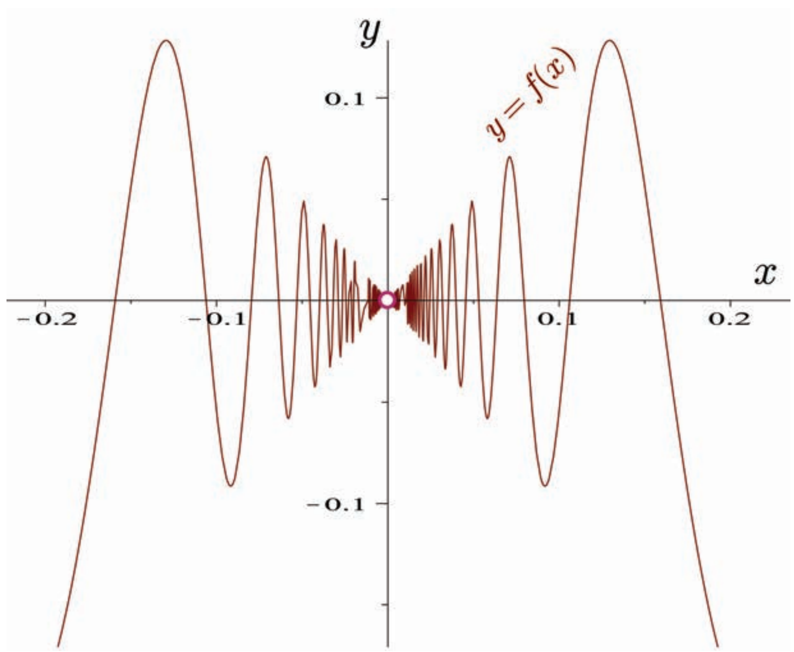
   T F $f(0) = 0$.

   T F $\lim_{x\to 0} f(x) = 0$.

   T F $\lim_{x\to 0} f(x) = \infty$.

   T F $\lim_{x\to \infty} f(x) = 0$.

   T F $\lim_{x\to -\infty} f(x) = \infty$.



2. *(9 points)* Let $f(x) = \dfrac{x+5}{x-2}$. Complete the nine blanks below using simple numerical values, showing how one computes $f'(3)$ from the definition.

$$\begin{aligned}
f'(3) &= \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{8+h}{1+h} - 8}{h} \\
&= \lim_{h\to 0} \frac{(8+h) - 8(1+h)}{h(1+h)} \\
&= \lim_{h\to 0} \frac{-7h}{h(1+h)} \\
&= \lim_{h\to 0} \frac{-7}{1+h} \\
&= -7.
\end{aligned}$$

3. *(12 points)* Multiple choice: Circle the correct response (A, B, C or D).

(a) The average rate of change of a function $f$ on the interval $[0,2]$ is

A. $f(2)-f(0)$

B. $\dfrac{f(2)-f(0)}{2}$

C. $\lim_{h\to 0} \dfrac{f(2+h)-f(h)}{h}$

D. $\lim_{h\to 0} \dfrac{f(2+h)-f(2)}{h}$

(b) The instantaneous rate of change of $f(x)$ at $x=2$ is

A. $\lim_{h\to 0} \dfrac{f(2+h)-f(h)}{h}$

B. $\lim_{h\to 0} \dfrac{f(2+h)-f(2)}{h}$

C. $\lim_{h\to 0} \dfrac{f(2)-f(h)}{2-h}$

D. $\lim_{h\to 0} \dfrac{f(2+h)-f(h)}{2}$

(c) If an object has position $s(t)$ at time $t$, then its average velocity during the time interval $[0,t]$ is

A. $s'(0)$

B. $\dfrac{s(t)-s(0)}{t-0}$

C. $\lim_{t\to 0} s(t)$

D. $\lim_{h\to 0} \dfrac{s(t+h)-s(h)}{h}$

(d) If an object has position $s(t)$ at time $t$, then its instantaneous velocity at time $t=0$ is

A. $s'(0)$

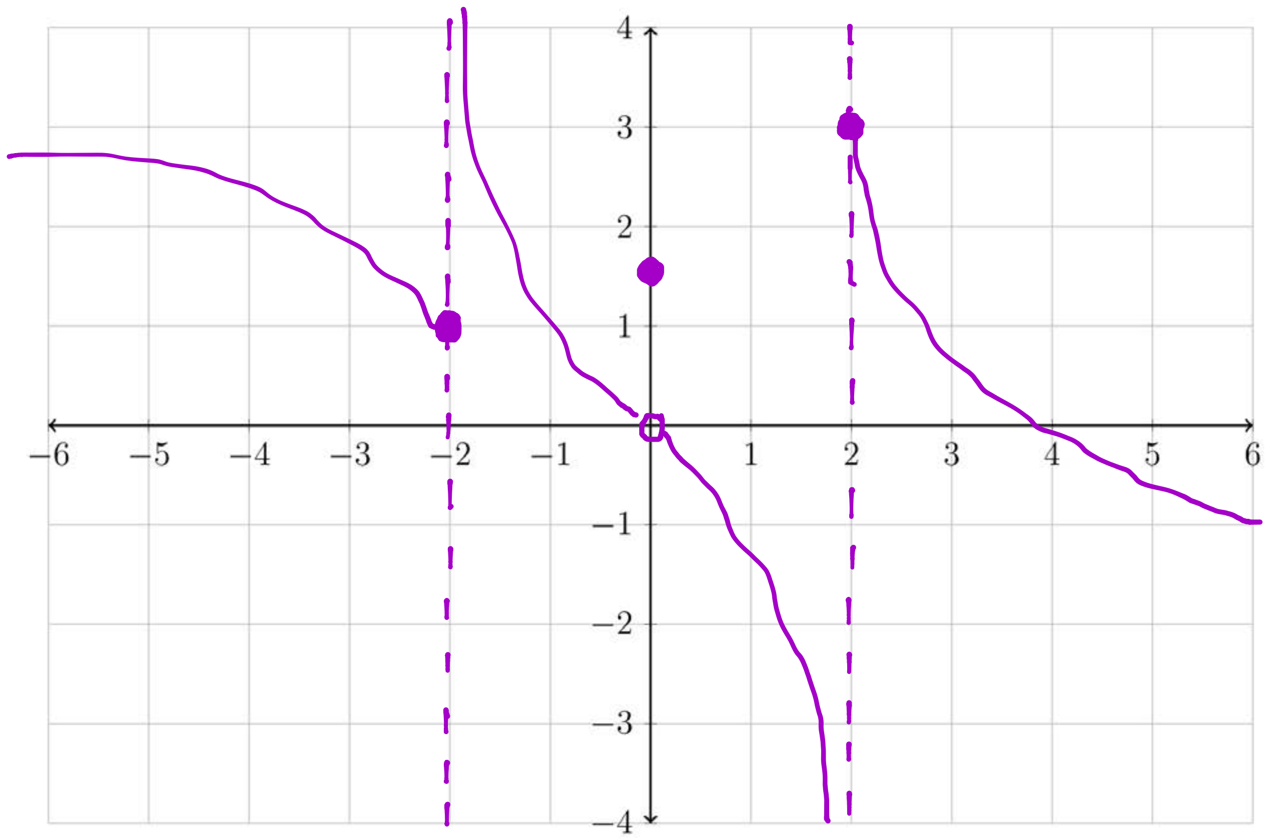
B. $\dfrac{s(t)-s(0)}{t-0}$

C. $\lim_{t\to 0} s(t)$

D. $\lim_{h\to 0} \dfrac{s(t+h)-s(h)}{h}$

4. *(12 points)* Using the axes below, sketch the graph of a function $g(x)$ satisfying the following conditions:

- $\lim_{x \to 2^+} g(x) = g(2) = 3$;
- $\lim_{x \to 2^-} g(x) = -\infty$;
- $\lim_{x \to -2^+} g(x) = \infty$;
- $\lim_{x \to -2^-} g(x) = g(-2) = 1$;
- $g(0)$ and $\lim_{x \to 0} g(x)$ are both defined, but $g(x)$ is not continuous at $x = 0$.



5. *(8 points)* The number of people in a theater at time $t$ (in minutes) is given by a function $f(t)$. Values, recorded at 2-minute intervals, are tabulated as shown:

| $t$ | 0 | 2 | 4 | 6 | 8 |
|---|---|---|---|---|---|
| $f(t)$ | 5 | 7 | 25 | 61 | 117 |

(a) During the entire 8-minute interval, what was the average rate of increase of people in the theater (in people per minute)?

$$\frac{f(8)-f(0)}{8-0} = \frac{117-5}{8-0} = 14 \text{ people per minute}$$

(b) Give a reasonable estimate for the instantaneous rate at which people were entering the theater at time $t = 5$ minutes.

$$\frac{f(6)-f(4)}{6-4} = \frac{61-25}{2} = 18 \text{ people per minute}$$

6. *(8 points)*

(a) The charge for parking a passenger car at the airport during a time interval of $t$ minutes, is $C(t)$ dollars. One driver who parks for 15 minutes is charged 5 dollars, so $C(15) = 5$. Another driver parks for 2 hours and is charged 20 dollars, so $C(120) = 20$.

Based on the information given, can we reasonably conclude that there is a time interval for which a driver will be charged exactly 12 dollars for parking? *Explain.*

No, there is no reason to expect that there is any time interval which would incur a 12 dollar charge for parking. The point is that the cost $C(t)$ is typically not a continuous function of time $t$, so the hypotheses are not satisfied.

(b) The temperature in a garage at time $t$ (in hours after midnight) on a given day is $T(t)$. We are given that the temperature is 25 degrees at 5am and 35 degrees at 11am, so $T(5) = 25$ and $T(11) = 35$.

Based on the information given, can we reasonably conclude that the temperature in the garage was 30 degrees at some time that morning? *Explain.*

Yes, assuming $T(t)$ is a continuous function, as is reasonable to assume. Since $T(5) < 30 < T(11)$, by the Intermediate Value Theorem, the temperature must have been exactly 30 degrees at some moment of time between 5 am and 11 am.

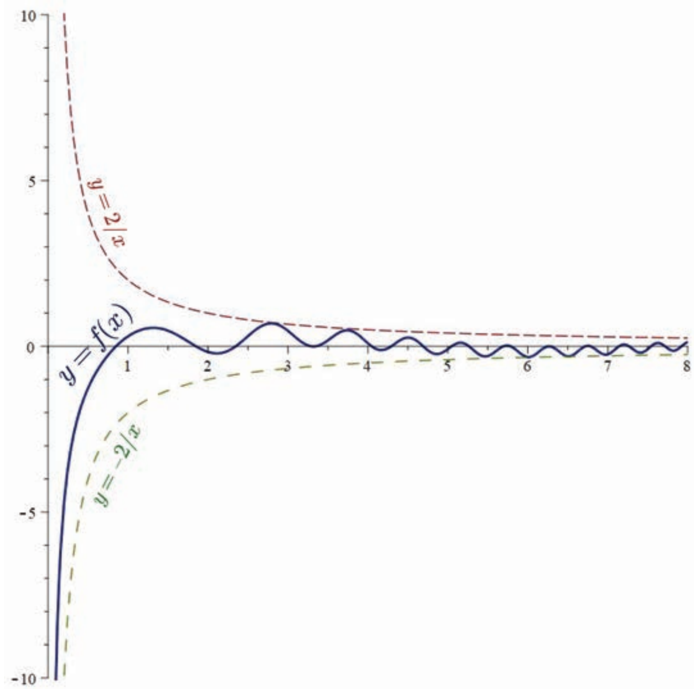
(c) *Name the theorem*, studied in class this semester, which describes situations under which we may draw conclusions such as those described above.

the Intermediate Value Theorem

7. *(8 points)* A function $f(x)$ is known to satisfy

$$-\frac{2}{x} \leqslant f(x) \leqslant \frac{2}{x}$$

for all $x > 0$. *One possible function* $f$ satisfying these inequalities is shown:




(a) *Using only the inequalities above*, can we determine $\lim_{x\to\infty} f(x)$ using the Squeeze Theorem? Explain, and if the limit is known, indicate its value.

Yes; $\lim_{x\to\infty} f(x) = 0$ by the Squeeze Theorem using the inequalities given, together with $\lim_{x\to\infty} \frac{2}{x} = 0 = \lim_{x\to\infty} \left(-\frac{2}{x}\right)$.

(b) *Using only the inequalities above*, can we determine $\lim_{x\to 0^+} f(x)$ using the Squeeze Theorem? Explain, and if the limit is known, indicate its value.

No; since $\lim_{x\to 0^+} \frac{2}{x} \neq \lim_{x\to 0^+} \left(\frac{-2}{x}\right)$, the hypotheses of the Squeeze Theorem are not satisfied and so we cannot determine $\lim_{x\to 0^+} f(x)$ based only on the information given.

8. *(9 points)* A function $f$ is given by

$$f(x) = \begin{cases} 2^x, & \text{for } x < 1; \\ a, & \text{for } x = 1; \text{ and} \\ \sqrt{b+x}, & \text{for } x > 1 \end{cases}$$

where $a$ and $b$ are constants.

Determine values of the constants $a$ and $b$ such that $f$ is continuous everywhere. Explain your work.

We have

$$\begin{cases} \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2^x = 2\ ; \\ f(1) = a\ ; \\ \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \sqrt{b+x} = \sqrt{b+1}\,. \end{cases}$$

In order for $f$ to be continuous at 1, we require these three values to coincide. This can only happen if $a=2$ and $b=3$.

9. *(12 points)* Find the following limits algebraically, using the limit laws (*not using calculator estimates!*). Use proper mathematical notation, symbols, syntax, and terminology at all times.

(a) $\lim_{x \to 0} \frac{x^3 - 8}{x^2 - 4}$ $= \frac{-8}{-4} = 2.$

(b) $\lim_{x \to 2} \frac{x^3 - 8}{x^2 - 4}$ $= \lim_{x \to 2} \frac{(x-2)(x^2+2x+4)}{(x-2)(x+2)} = \lim_{x \to 2} \frac{x^2+2x+4}{x+2} = \frac{12}{4} = 3.$

(c) $\lim_{x \to \infty} \frac{x^3 - 8}{x^2 - 4}$ $= \lim_{x \to \infty} \frac{x\left(1 - \frac{8}{x^3}\right)}{1 - \frac{4}{x^2}} = \infty$

Since $\lim_{x \to \infty} x = \infty$ and $\lim_{x \to \infty} \frac{1 - \frac{8}{x^3}}{1 - \frac{4}{x^2}} = \frac{1-0}{1-0} = 1.$

(d) $\lim_{x \to -\infty} \frac{x^3 - 8}{x^2 - 4}$ $= \lim_{x \to -\infty} \frac{x\left(1 - \frac{8}{x^3}\right)}{1 - \frac{4}{x^2}} = -\infty$

Since $\lim_{x \to -\infty} x = -\infty$ and $\lim_{x \to -\infty} \frac{1 - \frac{8}{x^3}}{1 - \frac{4}{x^2}} = \frac{1-0}{1-0} = 1.$

10. *(12 points)* Experiment suggests that a falling body will fall a distance of $s(t) = 16t^2$ feet in $t$ seconds.

(a) How far will it fall between $t = 2$ and $t = 3$?

$s(3) - s(2) = 144 - 64 = 80$ feet

(b) What is the average velocity on the interval $2 \leqslant t \leqslant 3$?

$$\frac{s(3) - s(2)}{3 - 2} = 80 \text{ ft/sec}$$

(c) What is the average velocity on the interval $2 \leqslant t \leqslant 2 + h$? (Here $h$ is a small positive number.)

$$\frac{s(2+h) - s(2)}{h} = \frac{64 + 64h + 16 - 16}{h} = 64 + h \text{ ft/sec}$$

(d) Find its instantaneous velocity at $t = 2$.

$$s'(2) = \lim_{h \to 0} \frac{s(2+h) - s(2)}{h} = \lim_{h \to 0} (64 + h) = 64 \text{ ft/sec}$$

(SCRATCH WORK)

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*For instructors' use only:*

| Question | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Points | 10 | 9 | 12 | 12 | 8 | 8 | 8 | 9 | 12 | 12 | 100 |
| Score | | | | | | | | | | | |