Math 2200-01 (Calculus I) Spring 2020

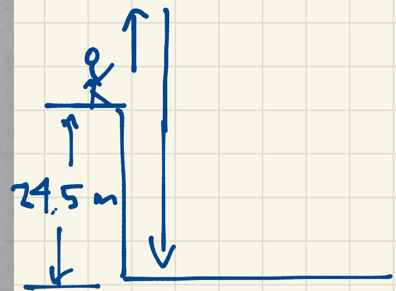
Book 2

Sec 3.6 #24.

Mar 2

Suppose a stone is thrown vertically upward from the edge of a cliff on Earth with an initial velocity of 19.6 m/s from a height of 24.5 m above the ground. The height (in meters) of the stone above the ground t seconds after it is thrown is $s(t) = -4.9t^2 + 19.6t + 24.5$.

a. Determine the velocity v of the stone after t seconds.
b. When does the stone reach its highest point?
c. What is the height of the stone at the highest point?
d. When does the stone strike the ground?
e. With what velocity does the stone strike the ground?
f. On what intervals is the speed increasing?



$s(t) = -4.9t^2 + 19.6t + 24.5, \quad 0 \le t \le 5.$
height of the stone above the ground in meters, at time t (in seconds).

(a) $v(t) = s'(t) = -9.8t + 19.6, \quad 0 \le t < 5.$
velocity at time t (in m/sec).

Note: $s(0) = 24.5$ m is the initial height;
$v(0) = 19.6$ m/sec is the initial velocity. In this problem, the motion is vertical with the positive direction being upwards.

(b) The stone reaches its highest point at the moment when the velocity changes sign from positive (upwards) to negative (downwards). At this moment the instantaneous velocity is zero. Solve $v(t) = -9.8t + 19.6 = 0$ to find $t = 2$ sec.

(c) The maximum height is $s(2) = 44.1$ m.

(d) The stone strikes the ground when $s(t) = -4.9t^2 + 19.6t + 24.5 = 0 = -4.9(t^2 - 4t - 5)$
$= -4.9(t-5)(t+1)$

This has two roots $t = -1, 5$ sec. But since $t \geq 0$, we must have $t = 5$ sec as the time when the stone hits the ground.

(e) The stone hits the ground with velocity $v(5) = -29.4$ m/sec (i.e. downwards at a speed of 29.4 m/sec).

(f) Speed is increasing during the time interval $2 < t < 5$ seconds.

<u>Remark</u> $a(t) = v'(t) = s''(t) = -9.8 \text{ m/sec}^2$ is constant.

<u>Sec 3.7 Chain Rule</u> Eg. find $\frac{d}{dx} \sin(e^x)$.

In general if $f(x) = g(h(x))$ and we know g', h', how do we find f'?

In other words, if

$$x \xmapsto{h} u \xmapsto{g} y$$

y: dependent variable

x: independent variable

u: intermediate variable

As an example, think of $u = e^x$, $y = \sin u$

Small changes Δx in x give rise to small changes Δu in u, giving small changes Δy in y.

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}.$$

This refers to average rates of change. To get instantaneous rates of change, let $\Delta x \to 0$ so $\Delta u \to 0$ and $\Delta y \to 0$ giving

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Mar 3

Eg. $\frac{d}{dx}\sin(e^x) = e^x\cos(e^x)$

$$x \longmapsto u = e^x \longmapsto y = \sin u = \sin(e^x)$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \underbrace{\cos u}_{\frac{dy}{du}} \cdot \underbrace{e^x}_{\frac{du}{dx}} = e^x\cos(e^x)$$

Eg. $\frac{d}{dx}(x^2+1)^3 = \frac{d}{dx}(x^6+3x^4+3x^2+1) = 6x^5+12x^3+6x$ (OLD WAY)

$\frac{d}{dx}(x^2+1)^3 = 3(x^2+1)^2\cdot 2x = 6x(x^2+1)^2$ (CHAIN RULE - NEW WAY)

CHECK: $6x(x^2+1)^2 = 6x(x^4+2x^2+1) = 6x^5+12x^3+6x$

agrees!

Rewriting this in function notation:

$$f(x) = g(h(x))$$

$$\begin{tikzcd} x \arrow[r, "h"] \arrow[rr, bend right, "f"'] & u = x^2+1 \arrow[r, "g"] & y = u^3 = (x^2+1)^3 \end{tikzcd}$$

$$f'(x) = \underbrace{3(x^2+1)^2}_{g'(u)}\cdot\underbrace{2x}_{h'(x)} = g'(h(x))\,h'(x)$$

$h(x) = x^2+1$

$h'(x) = 2x$

$g(u) = u^3$

$g'(u) = 3u^2$

26. Derivatives using tables Let $h(x) = f(g(x))$ and $k(x) = g(g(x))$. Use the table to compute the following derivatives.

a. $h'(1)$ **b.** $h'(2)$ **c.** $h'(3)$ **d.** $k'(3)$ **e.** $k'(1)$ **f.** $k'(5)$

x	1	2	3	4	5
$f'(x)$	-6	-3	8	7	2
$g(x)$	4	1	5	2	3
$g'(x)$	9	7	3	-1	-5

(a) $h(x) = f(g(x))$

$h'(x) = f'(g(x))\,g'(x)$

$h'(1) = f'(g(1))\,g'(1)$

$= f'(4)\cdot 9 = 7\cdot 9 = 63.$

(b) $h'(2) = f'(g(2))\,g'(2)$

$= f'(1)\cdot 7 = (-6)\cdot 7 = -42$

(c) $h'(3) = f'(g(3))\,g'(3)$

$= f'(5)\cdot 3 = 2\cdot 3 = 6$

(d) $k'(3) = g'(g(3))\,g'(3)$

$= g'(5)\cdot 3$

$= -5\cdot 3 = -15$

(e) $k(x) = g(g(x))$

$k'(x) = g'(g(x))\,g'(x)$

$k'(1) = g'(g(1))\,g'(1)$

$= g'(4)\cdot 9 = -1\cdot 9 = -9$

(f) $k'(5) = g'(g(5))\,g'(5)$

$= g'(3)(-5) = 3\cdot(-5) = -15.$

Eg. $\frac{d}{dx}\sin^2 x = \frac{d}{dx}(\sin x)(\sin x) = \sin x\cos x + \cos x\sin x = 2\sin x\cos x$ (OLD WAY)

$\frac{d}{dx}\sin^2 x = \frac{d}{dx}(\sin x)^2 = 2\sin x\cos x$ (NEW WAY - CHAIN RULE)

$$\frac{d}{dx}\sin\left(\frac{3x}{x^2+1}\right) = \cos\left(\frac{3x}{x^2+1}\right)\cdot\frac{(x^2+1)\cdot 3 - 3x(2x)}{(x^2+1)^2} = \frac{-3x^2+3}{(x^2+1)^2}\cos\left(\frac{3x}{x^2+1}\right)$$

$$= 3\,\frac{1-x^2}{(x^2+1)^2}\cos\left(\frac{3x}{x^2+1}\right)$$

Mar 4

$x \longmapsto u \longmapsto y \qquad \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$

$x \longmapsto u \longmapsto v \longmapsto y \qquad \frac{dy}{dx} = \frac{dy}{dv}\cdot\frac{dv}{du}\cdot\frac{du}{dx}$

Eg. $\frac{d}{dx}\sqrt{\tan(x^2)}$

$x \longmapsto u = x^2 \longmapsto v = \tan u \longmapsto y = \sqrt{v}$

Note: $\frac{d}{dv}\sqrt{v} = \frac{d}{dv} v^{1/2} = \frac{1}{2} v^{-\frac{1}{2}} = \frac{1}{2\sqrt{v}}$

$\frac{du}{dx} = 2x \qquad \frac{dv}{du} = \sec^2 u \qquad \frac{dy}{dv} = \frac{1}{2\sqrt{v}}$

$$\frac{dy}{dx} = \frac{dy}{dv}\cdot\frac{dv}{du}\cdot\frac{du}{dx} = \frac{1}{2\sqrt{v}}\cdot\sec^2 u\cdot 2x = \frac{2x\sec^2(x^2)}{2\sqrt{\tan(x^2)}} = \frac{x\sec^2(x^2)}{\sqrt{\tan(x^2)}}$$

OR: $\frac{d}{dx}\sqrt{\tan(x^2)} = \frac{1}{2\sqrt{\tan(x^2)}}\cdot\sec^2(x^2)\cdot 2x = \frac{x\sec^2(x^2)}{\sqrt{\tan(x^2)}}$

If $f(x) = \sec(3x+1)$, find $f'(x)$ and $f''(x)$.

Recall: $\frac{d}{dt}\sec t = \sec t \tan t$

$\frac{d}{dt}\tan t = \sec^2 t$

$$f'(x) = \sec(3x+1)\tan(3x+1)\cdot 3 = 3\sec(3x+1)\tan(3x+1)$$

$$f''(x) = 3\underbrace{\left(3\sec(3x+1)\tan(3x+1)\right)}_{\frac{d}{dx}\sec(3x+1)}\tan(3x+1) + 3\sec(3x+1)\left(\underbrace{\sec^2(3x+1)\cdot 3}_{\frac{d}{dx}\tan(3x+1)}\right)$$

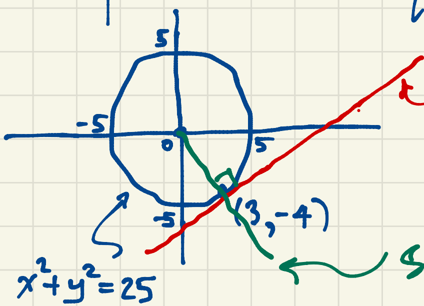
$$= 9\sec(3x+1)\tan^2(3x+1) + 9\sec^3(3x+1).$$

$\sec^2 t = 1 + \tan^2 t$

$\frac{d}{dx}\tan(3x+1) = \sec^2(3x+1)\cdot 3$

Sec 3.8: Implicit Differentiation

Example: Find the equation of the tangent line to the circle $x^2+y^2=25$ at the point $(3,-4)$.



slope $= \frac{3}{4}$

slope $= -\frac{4}{3}$

the normal line to the circle at $(3,-4)$ (ie. perpendicular) is $y = -\frac{4}{3}x$.

Take $\frac{d}{dx}$ of both sides: $\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}25$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\left.\frac{dy}{dx}\right|_{(3,-4)} = \frac{3}{4}$$

$$\frac{dy^2}{dx} = \frac{dy^2}{dy}\cdot\frac{dy}{dx} = 2y\frac{dy}{dx}$$

Mar 6

The tangent line to the circle at $(3,-4)$ is

$y+4 = \frac{3}{4}(x-3)$ i.e. $y = \frac{3}{4}x - \frac{25}{4}$

Alternatively, the circle $x^2+y^2=25$ has $y=\pm\sqrt{25-x^2}$ so it consists of an upper semicircle $y=\sqrt{25-x^2}$ and a lower semicircle $y=-\sqrt{25-x^2}$. Our point $(3,-4)$ is on the lower semicircle. If $f(x) = -\sqrt{25-x^2} = -(25-x^2)^{1/2}$ then $f'(x) = -\frac{1}{2}(25-x^2)^{-1/2}(-2x)$

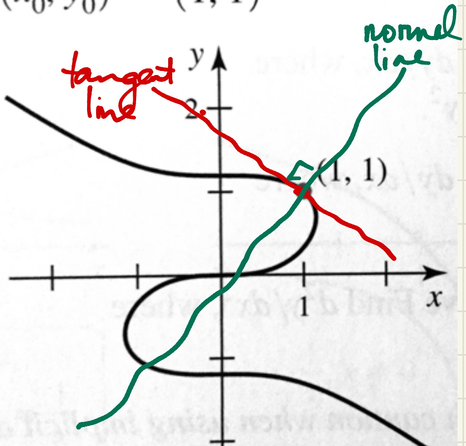
$= \dfrac{x}{\sqrt{25-x^2}}$ So $f'(3) = \dfrac{3}{\sqrt{25-9}} = \dfrac{3}{4}$. So the tangent line is $y = \frac{3}{4}x - \frac{25}{4}$ as before.

79–82. Visualizing tangent and normal lines

a. *Determine an equation of the tangent line and the normal line at the given point* (x_0, y_0) *on the following curves. (See instructions for Exercises 73–78.)*

b. *Graph the tangent and normal lines on the given graph.*

79. $3x^3 + 7y^3 = 10y;$
$(x_0, y_0) = (1, 1)$



Check first that $(1,1)$ lies on the curve!

$3\cdot1^3+7\cdot1^3=10\cdot1.$

$$\frac{d}{dx}(3x^3+7y^3) = \frac{d}{dx}10y$$

$$9x^2+21y^2\frac{dy}{dx} = 10\frac{dy}{dx}$$

Substitute $(1,1)$:

$m = \left.\frac{dy}{dx}\right|_{(1,1)}$ satisfies

$$9+21m = 10m$$
$$9 = -11m$$
$$m = -\frac{9}{11}$$

The tangent line to the curve at $(1,1)$ is

$$y-1 = -\frac{9}{11}(x-1)$$

ie. $y = -\frac{9}{11}x + \frac{20}{11}$

The normal line at $(1,1)$ has slope $-\frac{1}{m} = \frac{11}{9}$ so the normal line at $(1,1)$ is

$$y-1 = \frac{11}{9}(x-1) \quad \text{i.e.} \quad y = \frac{11}{9}x - \frac{2}{9}.$$

The power rule $\frac{d}{dx}x^n = nx^{n-1}$ was first explained for $n=0,1,2,3,\dots$ but it works for arbitrary exponent. Here's why:

$$\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}$$

If $n=-1,-2,-3,-4,\dots$ then $-n=1,2,3,4,\dots$

$$\frac{d}{dx}x^n = \frac{d}{dx}\frac{1}{x^{-n}} = \frac{x^{-n}\cdot 0 - 1\cdot(-n)x^{-n-1}}{(x^{-n})^2} = n\frac{x^{-n-1}}{x^{-2n}} = nx^{n-1}$$

If $n=\frac{a}{b}$ where a,b are integers and $y=x^n=x^{a/b}$ so $y^b=x^a$ so

$$by^{b-1}\frac{dy}{dx} = ax^{a-1} \text{ and } \frac{dy}{dx} = \frac{a}{b}\frac{x^{a-1}}{y^{b-1}} = \frac{a}{b}\frac{x^{a-1}}{x^{\frac{a}{b}(b-1)}} = \frac{a}{b}x^{\frac{a}{b}-1} = nx^{n-1}$$

Mar 9

p.207 #64. $x+y^3-y=1$. Find vertical tangent lines. Note: Points with horizontal tangent line have $\frac{dy}{dx}=0$; points with vertical tangent line have $\frac{dy}{dx}$ undefined.

$$\frac{d}{dx}(x+y^3-y) = \frac{d}{dx}1$$

$$1+3y^2\frac{dy}{dx}-\frac{dy}{dx} = 0$$

$$(3y^2-1)\frac{dy}{dx} = -1$$

$$\frac{dy}{dx} = \frac{1}{1-3y^2}$$

(b) Since $\frac{dy}{dx}\neq 0$, the curve has no horizontal tangent lines.

(a) For vertical tangent lines, $\frac{dy}{dx}$ is undefined so $y=\pm\frac{1}{\sqrt{3}}$.

$$x = 1+y-y^3 = 1+y(1-y^2) = 1\pm\frac{1}{\sqrt{3}}\left(1-\frac{1}{3}\right) = 1\pm\frac{2}{3\sqrt{3}}$$

The curve $x + y^3 - y = 1$ has vertical tangent lines at $\left(1+\frac{2}{3\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and at $\left(1-\frac{2}{3\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$. These are the lines $x = 1+\frac{2}{3\sqrt{3}}$ and $x = 1-\frac{2}{3\sqrt{3}}$ respectively.

p. 206 #53. Find $\frac{d^2y}{dx^2}$.

$$x + y = \sin y$$

$$\frac{d}{dx}(x+y) = \frac{d}{dx}\sin y$$

$$1 + \frac{dy}{dx} = (\cos y)\frac{dy}{dx}$$

There are now two ways to proceed to find $\frac{d^2y}{dx^2}$. One way: solve

$$\frac{dy}{dx} = \frac{1}{\cos y - 1}$$

$$\frac{d^2y}{dx^2} = \frac{(\cos y - 1)\cdot 0 - 1\cdot(-\sin y)\frac{dy}{dx}}{(\cos y - 1)^2} = \frac{\sin y}{(\cos y - 1)}\cdot\frac{dy}{dx}$$

$$= \frac{\sin y}{(\cos y - 1)^3}$$

The other way:

$$\frac{d}{dx}\left(1 + \frac{dy}{dx}\right) = \frac{d}{dx}\left(\cos y\,\frac{dy}{dx}\right)$$

$$\frac{d^2y}{dx^2} = \cos y\,\frac{d^2y}{dx^2} + \frac{dy}{dx}(-\sin y)\frac{dy}{dx} = \cos y\,\frac{d^2y}{dx^2} - \sin y\left(\frac{dy}{dx}\right)^2$$

$$(1 - \cos y)\frac{d^2y}{dx^2} = -\sin y\left(\frac{dy}{dx}\right)^2 = \frac{-\sin y}{(\cos y - 1)^2} \Rightarrow \frac{d^2y}{dx^2} = \frac{\sin y}{(\cos y - 1)^3}$$

Sec 3.9 Derivatives of Exponential and Logarithmic Functions

$$\frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx} e^{cx} = ce^{cx}$$

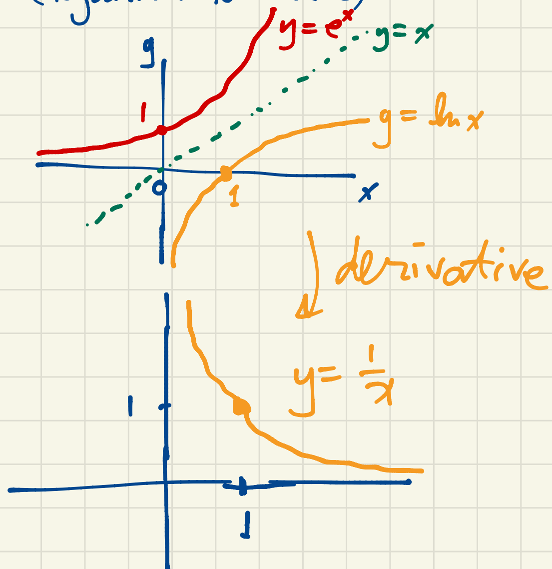
$$\frac{d^2}{dx^2} e^{cx} = c^2 e^{cx}$$

$$\frac{d^3}{dx^3} e^{cx} = c^3 e^{cx}$$

Natural logarithm $y = \ln x \iff x = e^y$

(logarithm to base e)

$$\begin{tikzcd} x \arrow[r, bend left, "\ln"] & y \arrow[l, bend left, "\exp"] \end{tikzcd}$$



If $y = \ln x$, find $\frac{dy}{dx}$.

$\iff x = e^y$

$$\frac{d}{dx} x = \frac{d}{dx} e^y$$

$$1 = e^y \cdot \frac{dy}{dx}$$

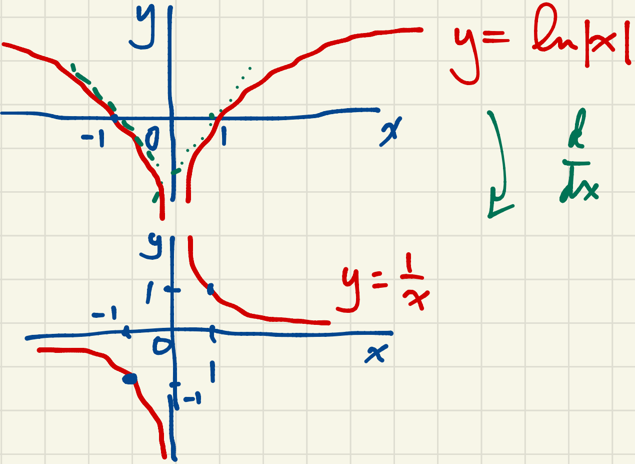
$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

Mar 10

$$\frac{d}{dx} \ln(3x) = \frac{1}{3x} \cdot 3 = \frac{1}{x}$$

$$\frac{d}{dx} \ln(3x) = \frac{d}{dx} (\ln 3 + \ln x) = 0 + \frac{1}{x} = \frac{1}{x}$$



$$\frac{d}{dx}\ln|x| = \frac{1}{x}, \quad x \neq 0.$$

Check: If $x>0$, $\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln x = \frac{1}{x}$.

If $x<0$ then $|x| = -x$ so

$$\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln(-x) = \frac{1}{-x}\cdot(-1) = \frac{1}{x}.$$

The inverse of $f(x) = a^x$ (exponential with base $a>0$) is

$$g(x) = f^{-1}(x) = \log_a x.$$

$$x = a^y = (e^{\ln a})^y = e^{(\ln a)y}$$

$$\iff \ln x = (\ln a)y$$

$$\iff y = \frac{\ln x}{\ln a}$$

NOTE: $\boxed{\log_a x = \frac{\ln x}{\ln a}}$

i.e.

$$a^y = x \underset{\text{exponential}}{\overset{\text{logarithm}}{\underset{f}{\overset{g}{\rightleftarrows}}}} y = \log_a x$$

$$\frac{d}{dx}a^x = a^x\ln a \; ; \quad \frac{d}{dx}\log_a x = \frac{1}{(\ln a)x} = \frac{1}{x\ln a}$$

$$\frac{d}{dx}e^x = e^x \; ; \quad \frac{d}{dx}\ln x = \frac{1}{x}.$$

Eg. $$\frac{d}{dx}\frac{x^2e^x}{\sqrt{x^2+1}} = \frac{\sqrt{x^2+1}\,(2xe^x+x^2e^x) - x^2e^x\left(\frac{1}{2\sqrt{x^2+1}}\cdot 2x\right)}{x^2+1}$$

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

$$= \frac{(x^2+1)(2xe^x+x^2e^x) - x^3e^x}{(x^2+1)^{3/2}}$$

$$= \frac{(x^4+2x^3+x^2+2x-x^3)e^x}{(x^2+1)^{3/2}} = \frac{(x^4+x^3+x^2+2x)e^x}{(x^2+1)^{3/2}}$$

Logarithmic Differentiation

Alternatively: Take ln on both sides of $y = \dfrac{x^2e^x}{\sqrt{x^2+1}}$ to get

$$\ln y = 2\ln x + x - \tfrac{1}{2}\ln(x^2+1)$$

using $\ln(ab) = \ln a + \ln b$

$\ln a^k = k\ln a$

$\ln \frac{a}{b} = \ln a - \ln b$

$\ln e^x = x$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + 1 - \frac{1}{\cancel{2}(x^2+1)}\cdot\cancel{2}x = \frac{2x^2+2+x^3+x-x^2}{x(x^2+1)}$$

$$= \frac{x^3+x^2+x+2}{x(x^2+1)}$$

$$\frac{dy}{dx} = \frac{x^3+2x^2+2}{x(x^2+1)}y = \frac{x^3+x^2+x+2}{x(x^2+1)}\cdot\frac{x^2e^x}{\sqrt{x^2+1}} = \frac{(x^4+x^3+x^2+2x)e^x}{(x^2+1)^{3/2}}$$

$$\frac{d}{dx} x^7 = 7x^6$$

$$\frac{d}{dx} 5^x = 5^x \ln 5$$

$$\frac{d}{dx} x^x = x^x (1 + \ln x)$$

$y = x^x$ Use logarithmic differentiation.

$$\ln y = x \ln x$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = 1 \cdot \ln x + x \cdot \frac{1}{x} = 1 + \ln x$$

$$\frac{dy}{dx} = (1 + \ln x) y = (1 + \ln x) x^x$$

Mar 11

Sec 3.10 Derivatives of Inverse Functions

$$\begin{tikzcd} x \arrow[r, bend left, "g"] & y \arrow[l, bend left, "f"] \end{tikzcd}$$

$x = f(y)$, $y = g(x) = f^{-1}(x)$

$(f^{-1})' = g'$ can be expressed using $f(g(x)) = x$

$$f'(g(x))\, g'(x) = 1 \Rightarrow g'(x) = \frac{1}{f'(g(x))}$$

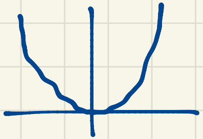
In differential notation,

$$x \xrightarrow{g} y \xrightarrow{f} x \qquad 1 = \frac{dx}{dx} = \frac{dx}{dy} \cdot \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{dx/dy}$$

Eg. $f(x) = x^2$ $f(3) = 9 = f(-3)$

$f^{-1}(9) = 3$? or -3 ?

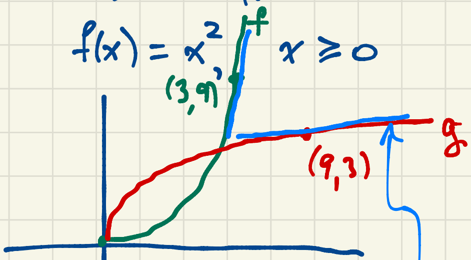
$3 \xrightarrow{f} 9$, $-3 \xrightarrow{f} 9$



This f doesn't have an inverse function

So restrict the domain:

$f(x) = x^2$, $x \geq 0$



$g(x) = f^{-1}(x) = \sqrt{x}$, $x \geq 0$

$f(x) = x^2$ $f'(3) = 6$

$f'(x) = 2x$

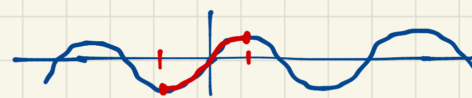
$$g'(9) = \frac{1}{f'(g(9))} = \frac{1}{2g(9)} = \frac{1}{2 \cdot 3} = \frac{1}{6}.$$

slope of the tangent line to the graph of g at $(9,3)$

tangent line

$y - 3 = \frac{1}{6}(x - 9)$

$y = \frac{1}{6}x + \frac{3}{2}$



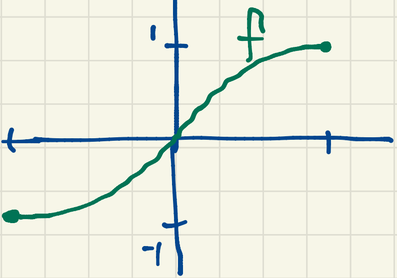
Similarly, the function $f(x) = \sin x$ does not have an inverse.

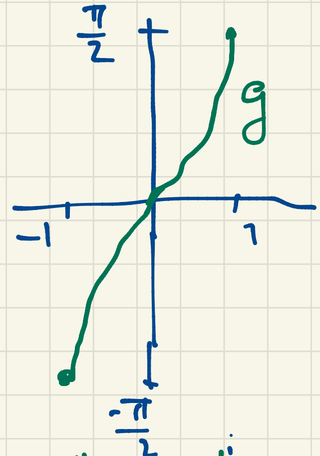
$0 \xrightarrow{f} 0$, $\pi \xrightarrow{f} 0$, $2\pi \to 0$

$f^{-1}(0) = 0$? π? 2π? 3π? $-\pi$?

So restrict f to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

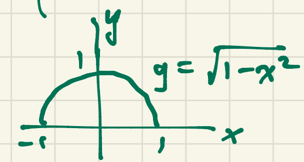
$$f(x) = \sin x, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$



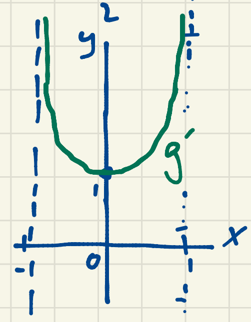


$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$$

Compare:



$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\cos(g(x))} = \frac{1}{\sqrt{1-x^2}}$$



$y = g(x)$

$x = f(y) = \sin y$

$\cos(g(x)) = \cos y$

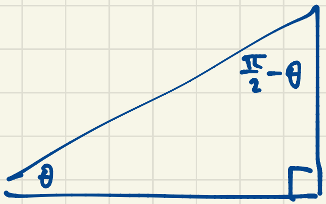
$\sin^2 y + \cos^2 y = 1$

$\cos y = \pm\sqrt{1-\sin^2 y}$

But for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $\cos y \ge 0$.

So $\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$

$$\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$$



$$\sin\theta = \cos\left(\frac{\pi}{2}-\theta\right) = x$$

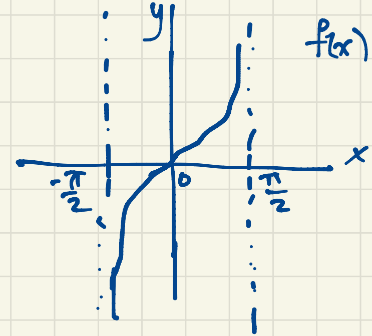
$$\theta = \sin^{-1}x$$

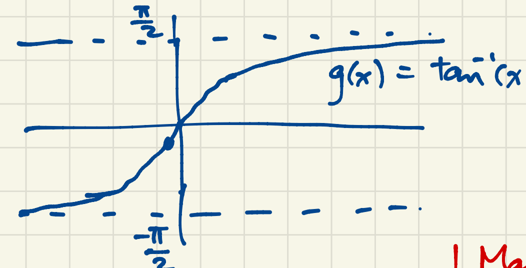
$$\frac{\pi}{2}-\theta = \cos^{-1}x$$

$$\frac{\pi}{2} = \sin^{-1}x + \cos^{-1}x$$

$$0 = \frac{d}{dx}\sin^{-1}x + \frac{d}{dx}\cos^{-1}x$$

$f(x) = \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$





$$\frac{d}{dx}\tan^{-1}x = \frac{1}{x^2+1} \qquad \text{Why?}$$

$y = \tan^{-1}x \iff x = \tan y$. Take $\frac{d}{dx}$ on both sides.

$$1 = (\sec^2 y)\frac{dy}{dx}$$

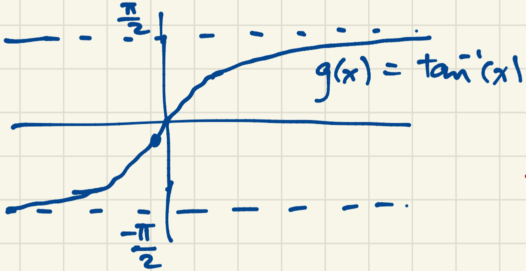
$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{x^2+1}$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{x^2+1}$$

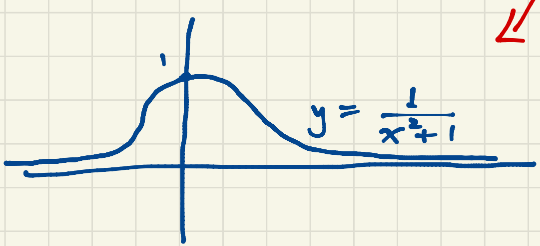
Mar 13

$$\frac{\sin^2 y + \cos^2 y}{\cos^2 y} = \frac{1}{\cos^2 y}$$

$$x^2+1 = \tan^2 y + 1 = \sec^2 y$$



derivative



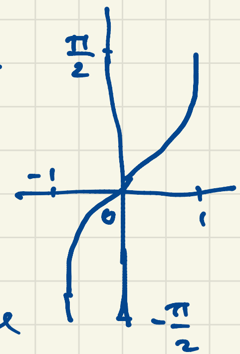
Warning

$$\sin^2 x = (\sin x)^2$$

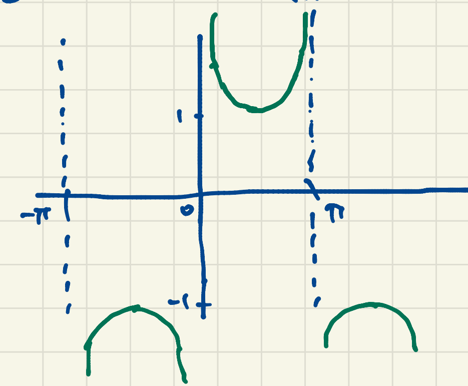
$$\cos^3 x = (\cos x)^3$$

$$\tan^{-1} x \neq (\tan x)^{-1} = \frac{1}{\tan x} = \cot x$$

$y = \sin^{-1} x$



$$y = (\sin x)^{-1} = \frac{1}{\sin x} = \csc x$$



Eg. Find the slope of the tangent line to the curve $xe^y + y = 2$ at $(2,0)$.

Check: $(2,0)$ lies on the curve. $2 \cdot e^0 + 0 = 2$.

$$xe^y \cdot y' + e^y + y' = 0$$

$$(1+xe^y)\, y' = -e^y$$

$$y' = -\frac{e^y}{1+xe^y}$$

The tangent line at $(2,0)$ has slope $\quad y' = \frac{dy}{dx}$

$$-\frac{e^y}{1+xe^y}\bigg|_{(2,0)} = -\frac{1}{1+2\cdot 1} = -\frac{1}{3}$$

The tangent line is

$$y - 0 = -\frac{1}{3}(x-2)$$

i.e. $y = -\frac{1}{3}x + \frac{2}{3}$

March 31

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{x^2+1}$$

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}, \quad |x|>1$$

p.235 #30. $f(t) = (\cos^{-1}t)^2$ NOT $\cos^{-2}t$

$$f'(t) = (2\cos^{-1}t)\cdot\left(-\frac{1}{\sqrt{1-t^2}}\right) = -\frac{2\cos^{-1}t}{\sqrt{1-t^2}}$$

#28. $f(t) = \ln(\sin^{-1}t^2)$

$$f'(t) = \frac{1}{\sin^{-1}t^2}\cdot\frac{1}{\sqrt{1-(t^2)^2}}\cdot 2t = \frac{2t}{(\sin^{-1}t^2)\sqrt{1-t^4}}$$

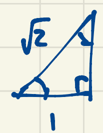
#24. $f(x) = \sec^{-1}\sqrt{x}, \quad x>1$

$$f'(x) = \frac{1}{|\sqrt{x}|\sqrt{(\sqrt{x})^2-1}}\cdot\frac{1}{2\sqrt{x}} = \frac{1}{2x\sqrt{x-1}}$$

$|\sqrt{x}|\sqrt{x} = x$ since $x>0$

$\sqrt{x^2} = |x|$

#41. Find the tangent line to the graph of $f(x) = \tan^{-1}2x$ at $\left(\frac{1}{2}, \frac{\pi}{4}\right)$. Check: $f\left(\frac{1}{2}\right) = \tan^{-1}1 = \frac{\pi}{4}$



$\tan\frac{\pi}{4} = \frac{1}{1} = 1$

$\tan^{-1}1 = \frac{\pi}{4}$

$$f'(x) = \frac{1}{(2x)^2+1}\cdot 2 = \frac{2}{4x^2+1}$$

$$f'\left(\tfrac{1}{2}\right) = \frac{2}{2} = 1$$

The tangent line is $y - \frac{\pi}{4} = 1\left(x - \frac{1}{2}\right)$

i.e. $y = x + \frac{\pi}{4} - \frac{1}{2}$.

Implicit Differentiation

Sec 3.8 #38. $\sin x \cos y = \sin x + \cos y$. Find $\frac{dy}{dx}$.

$$\frac{d}{dx}(\sin x \cos y) = \frac{d}{dx}(\sin x + \cos y)$$

$$\cos x \cos y + \sin x\left(-\sin y \frac{dy}{dx}\right) = \cos x - \sin y \frac{dy}{dx}$$

$$\cos x \cos y - \sin x \sin y \frac{dy}{dx} = \cos x - \sin y \frac{dy}{dx}$$

$$\sin y \frac{dy}{dx} - \sin x \sin y \frac{dy}{dx} = \cos x - \cos x \cos y$$

$$\sin y(1 - \sin x)\frac{dy}{dx} = \cos x(1 - \cos y)$$

$$\frac{dy}{dx} = \frac{(1-\sin x)\sin y}{(1-\cos y)\cos x}$$

$$\boxed{\begin{aligned}\frac{d\cos y}{dx} &= \frac{d\cos y}{dy} \cdot \frac{dy}{dx}\\ &= -\sin y \frac{dy}{dx}\end{aligned}}$$

p.205 #26. Find the slope of the tangent line to the curve $(x+y)^{2/3}=y$ at $(4,4)$.

Check: $(4+4)^{2/3} = 8^{2/3} = (8^{\frac{1}{3}})^2 = 2^2 = 4$ so $(4,4)$ lies on the curve.

$$\frac{d}{dx}(x+y)^{2/3} = \frac{d}{dx}y$$

$$\frac{2}{3}(x+y)^{-\frac{1}{3}}\left(1+\frac{dy}{dx}\right) = \frac{dy}{dx}$$

Let $m = \left.\frac{dy}{dx}\right|_{(4,4)}$.

$$\frac{2}{3}(8)^{-\frac{1}{3}}(1+m) = m$$

$$\frac{2}{3}\cdot\frac{1}{2}(1+m) = m$$

$$\frac{1}{3}(1+m) = m$$

$$1+m = 3m$$

$$1 = 2m$$

$$m = \frac{1}{2}$$

The tangent line at $(4,4)$ is

$$y-4 = \frac{1}{2}(x-4)$$

i.e. $y = \frac{1}{2}x + 2$.

Sec 3.5

$$\lim_{x\to 1} \frac{\sin x}{x} = 1$$

#19. $$\lim_{x\to 2} \frac{\sin(x-2)}{x^2-4} = \lim_{u\to 0} \frac{\sin u}{(u+2)^2-4} = \lim_{u\to 0} \frac{\sin u}{u^2+4u}$$

Substitute $u = x-2$

$x = u+2$

$$= \lim_{u\to 0} \frac{\sin u}{u(u+4)}$$

$$= \lim_{u\to 0} \left(\frac{\sin u}{u} \cdot \frac{1}{u+4}\right)$$

$$= 1 \cdot \frac{1}{4} = \frac{1}{4}$$

April 1

Sec 3.5 #14. $$\lim_{x\to 0} \frac{\tan 7x}{\sin x} = \lim_{x\to 0} \frac{(\tan 7x)/x}{(\sin x)/x}$$

$$= \frac{7}{1} = 7$$

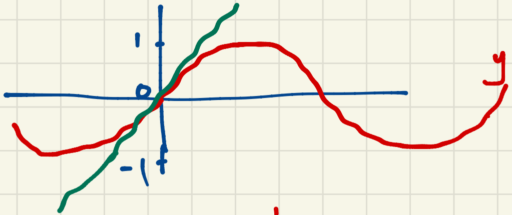
$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\sin x}{x \cos x} = \lim_{x\to 0} \left(\frac{\sin x}{x} \cdot \frac{1}{\cos x}\right) = 1 \cdot 1 = 1$$

$$\lim_{x\to 0} \frac{\tan 7x}{x} = \lim_{u\to 0} \frac{\tan u}{u/7} = 7 \lim_{u\to 0} \frac{\tan u}{u} = 7 \cdot 1 = 7.$$

$u = 7x$

$\frac{u}{7} = x$

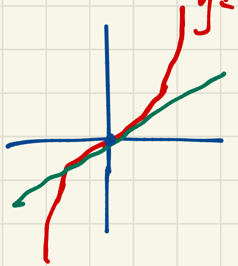


$y = \sin x$

$\lim_{x\to 0} \frac{\sin x}{x} = 1$ says for $x \approx 0$, $\sin x \approx x$,

$$\frac{\sin x}{x} \approx 1.$$

$y = \tan x$



$\lim_{x\to 0} \frac{\tan x}{x} = 1$ For $x \approx 0$, $\frac{\tan x}{x} \approx 1$, $\tan x \approx x$

$\tan 7x \approx 7x$

for $x \approx 0$, $\frac{\tan 7x}{\sin x} \approx \frac{7x}{x} = 7$.

In Calc II we'll see $\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots$

dominant term when $x \approx 0$.

Recall: $\lim_{x\to 0} \frac{\tan x}{x} = 1$ says $\left.\frac{d}{dx}\tan x\right|_{x=0} = 1$.

If $f(x) = \tan x$ then $f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{\tan(h) - 0}{h} = \lim_{h\to 0} \frac{\tan h}{h} = 1$

Sec 3.5 #60. $y = \frac{1}{2}e^x \cos x$

Use abbreviation $y' = \frac{dy}{dx}$, $y'' = \frac{d^2y}{dx^2}$ if you are careful to avoid ambiguity.

$$y' = \frac{1}{2}e^x \cos x + \frac{1}{2}e^x(-\sin x)$$

$$= \frac{1}{2}e^x(\cos x - \sin x)$$

$$y'' = \frac{1}{2}e^x(\cos x - \sin x) + \frac{1}{2}e^x(-\sin x - \cos x)$$

$$= \frac{1}{2}e^x(-2\sin x) = -e^x \sin x$$

Sec 3.9 Logarithmic Differentiation

$\frac{d}{dx} x^x = ?$

Not power rule $\frac{d}{dx} x^k = kx^{k-1}$ (k constant)

Not exponential rule $\frac{d}{dx} a^x = a^x \ln a$ (a constant)

Use logarithmic differentiation.

$$y = x^x$$

$$\ln y = \ln(x^x) = x \ln x$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln x)$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

$$\frac{dy}{dx} = (1 + \ln x)y = (1 + \ln x)x^x$$

$$\frac{d}{dx}(\ln x)^x =$$

$$y = (\ln x)^x$$

$$\ln y = \ln\left((\ln x)^x\right) = x\ln(\ln x)$$

$$\frac{d}{dx}\ln y = \frac{d}{dx}\, x\ln(\ln x)$$

$$\frac{1}{y}\frac{dy}{dx} = 1\cdot\ln(\ln x) + \cancel{x}\cdot\frac{1}{\ln x}\cdot\frac{1}{\cancel{x}}$$

$$= \ln(\ln x) + \frac{1}{\ln x}$$

$$\frac{dy}{dx} = \left[\ln\ln x + \frac{1}{\ln x}\right](\ln x)^x$$

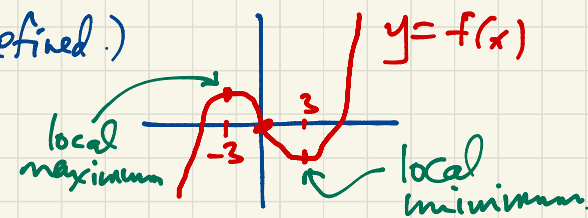
$$\ln(a^b) = b\ln a$$
$$\neq (\ln a)^b$$

$$\frac{d}{dx}\ln(\ln x) =$$
$$\frac{d\ln\ln x}{d\ln x}\cdot\frac{d\ln x}{dx}$$
$$= \frac{1}{\ln x}\cdot\frac{1}{x}$$

April 3

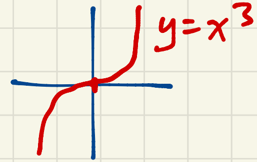
Sec 4.1 p.248 #25. Find the critical points of $f(x) = \frac{x^3}{3} - 9x$.

$f'(x) = x^2 - 9 = (x+3)(x-3)$. The critical points are ± 3. (Note: in this case there are no points where the derivative is undefined.)



Note: f has no maximum or minimum.
Only a local maximum point $(-3, 18)$ and a local minimum point $(3, -18)$.

Keep examples in mind eg.



has a critical point at the origin. $\left.\frac{dy}{dx}\right|_{(0,0)} = \left.3x^2\right|_{x=0} = 0$.

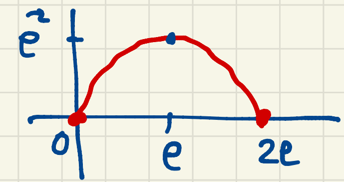
This function is increasing; it has no local (or global) maximum or minimum.

p.248 #48. $f(x) = 2ex - x^2$ on $[0, 2e]$. Note: $0 \le x \le 2e$.

By the way since we have a continuous function defined on a closed interval, it has a maximum and minimum. To find them check endpoints and critical points.

$f'(x) = 2e - 2x = 2(e-x)$. The only critical point is e. Note: e is in the interval $[0,2e]$

There are no points where f' is undefined.



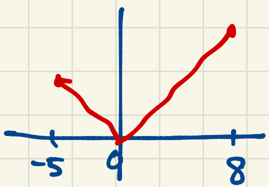
$$f(0) = 0$$
$$f(e) = 2e^2 - e^2 = e^2$$
$$f(2e) = 4e^2 - 4e^2 = 0.$$

So f has maximum value e^2 (attained at e) and minimum value 0 (attained at $0, 2e$).

Sometimes we refer to maximum point at 3, minimum points at $0, 2e$; other times we refer to (e, e^2) as the maximum point and $(0,0)$, $(2e, 0)$ as the minimum points. But always we refer to 0 and e^2 as the minimum and maximum value.

Eg. Find the maximum and minimum values of $f(x) = |x|$ on $[-5, 8]$.

Note: This function is continuous. The maximum point is $(8,8)$. The minimum point is $(0,0)$.



You can see this without calculus. If you use calculus,

$$f'(x) = \begin{cases} -1, & \text{if } -5 < x < 0; \\ 1, & \text{if } 0 < x < 8. \end{cases}$$

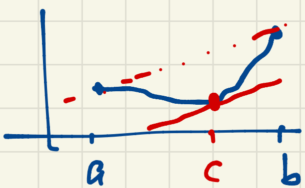
The only critical point is at 0.

$$\left.\begin{aligned} f(-5) &= 5 \\ f(0) &= 0 \\ f(8) &= 8 \end{aligned}\right\}$$

So the maximum value 8 occurs at 8; the minimum value 0 occurs at 0.

April 6

Mean Value theorem



p. 255 # 22. $f(x) = x^3 - 2x^2$ on $[0, 1]$.

Average rate of change of f on $[0,1]$ is

$$\frac{f(1) - f(0)}{1 - 0} = \frac{-1 - 0}{1 - 0} = -1$$

$f'(x) = 3x^2 - 4x$ Find $c \in (0,1)$ such that $f'(c) = 3c^2 - 4c = -1$

$3c^2 - 4c + 1 = 0$

$(3c - 1)(c - 1) = 0$

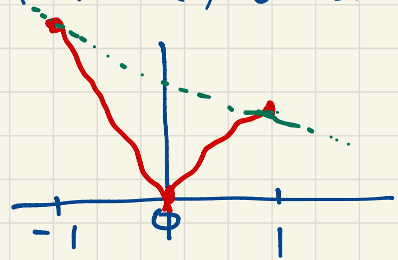
$c = \frac{1}{3}$ or 1

$\boxed{c = \frac{1}{3}}$

(a) Since f is a polynomial, it is continuous and differentiable everywhere. So the Mean Value Theorem applies.

(b) $c = \frac{1}{3}$ is the point (only one in this case) where $f'(c) = \dfrac{f(1) - f(0)}{1 - 0}$.

#23. $f(x) = \begin{cases} -2x & \text{if } x < 0; \\ x & \text{if } x \geq 0 \end{cases}$ on $[-1, 1]$.

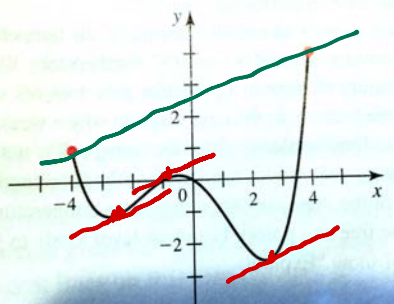


The average rate of change of f on $[-1, 1]$ is $\dfrac{f(1) - f(-1)}{1 - (-1)} = -\frac{1}{2}$

(a) No; this function is not differentiable on the interval $(-1, 1)$ (it is not differentiable at 0) so we cannot apply the Mean Value Theorem.

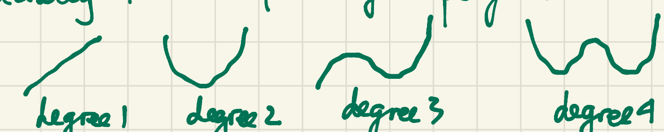
There is no point in $(-1, 1)$ where the tangent line has slope $-\frac{1}{2}$.

39. Mean Value Theorem and graphs By visual inspection, locate all points on the interval $(-4, 4)$ at which the slope of the tangent line equals the average rate of change of the function on the interval $[-4, 4]$.



There appear to be 3 solutions as shown.

Actually this is probably a polynomial of degree 4.



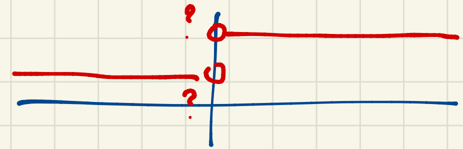
Solving $f'(c) = \square$ will then have 3 roots.

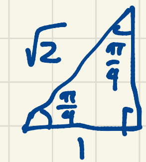
p.255 #34. (a) Use derivatives to show that $f(x) = \tan^{-1}x + \tan^{-1}\left(\frac{1}{x}\right)$ is constant for $x > 0$ and for $x < 0$.

To show f is constant, show $f' = 0$. (Theorem 4.5)

$$f'(x) = \frac{1}{x^2+1} + \frac{1}{\left(\frac{1}{x}\right)^2+1}\cdot\left(-\frac{1}{x^2}\right) = \frac{1}{x^2+1} - \frac{1}{1+x^2} = 0.$$

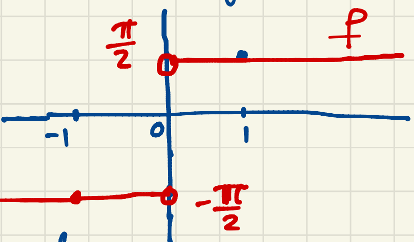
So the graph of f might look like





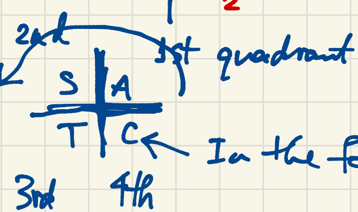
$\tan\frac{\pi}{4} = \frac{1}{1} = 1$

$\tan^{-1}1 = \frac{\pi}{4}$



(b) $f(1) = \tan^{-1}1 + \tan^{-1}1 = 2\tan^{-1}1$

$= 2\cdot\frac{\pi}{4} = \frac{\pi}{2}$

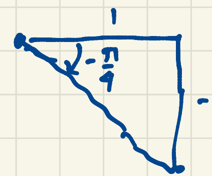


CAST rule

(c) $f(-1) = \tan^{-1}(-1) + \tan^{-1}(-1) = 2\tan^{-1}(-1)$

$= -\frac{\pi}{2}$

In the fourth quadrant, Cosine is positive; sine and tangent are negative.



$\tan\left(-\frac{\pi}{4}\right) = \frac{-1}{1}$

$= -1$

$\tan^{-1}(-1) = -\frac{\pi}{4}$

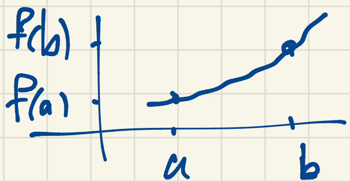
$$\tan^{-1}x + \tan^{-1}\frac{1}{x} = \begin{cases} \frac{\pi}{2}, & \text{if } x > 0; \\ -\frac{\pi}{2}, & \text{if } x < 0. \end{cases}$$

April 7

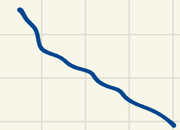
Sec 4.3

Increasing and Decreasing Functions.

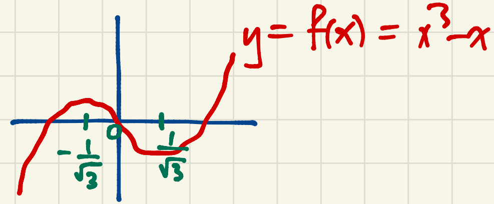
A function is <u>increasing</u> if when we move to the right on its graph, we go up.
(whenever $a < b$, we have $f(a) < f(b)$.)



<u>Decreasing</u>: $f(a) > f(b)$ whenever $a < b$.



eg. $f(x) = x^3 - x$



$f'(x) = 3x^2 - 1$

$= (\sqrt{3}\,x + 1)(\sqrt{3}\,x - 1)$

Critical points: $\pm \frac{1}{\sqrt{3}}$

At $-\frac{1}{\sqrt{3}}$, f has a local maximum.

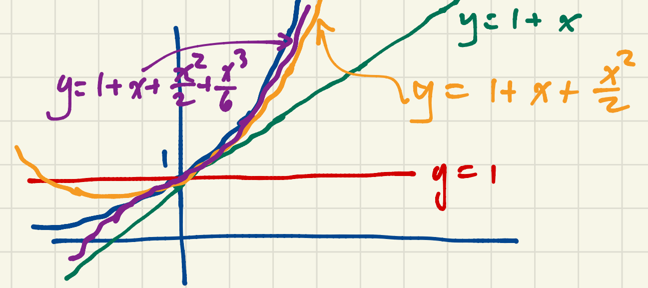
At $\frac{1}{\sqrt{3}}$, f has a local minimum.

On $(-\infty, -\frac{1}{\sqrt{3}})$, $f' > 0$ so f is increasing.

on $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, $f' < 0$ so f is decreasing

on $(\frac{1}{\sqrt{3}}, \infty)$, $f' > 0$ so f is increasing.

I will show you that for $x>0$, $e^x > \boxed{\boxed{\boxed{1+x} + \frac{x^2}{2}} + \frac{x^3}{6}}$.



First of all, $e^x > 1$ for $x>0$.

Why? Take $f(x)=e^x$. $f'(x)=e^x>0$ so f is increasing.

So $f(x) > f(0)$ whenever $x>0$.

i.e. $e^x > 1$ whenever $x>0$.

Now take $g(x) = e^x - (1+x)$. I want to show $g(x)>0$ whenever $x>0$.

$g'(x) = e^x - 1 > 0$ for $x>0$. So g is increasing. So $g(x) > g(0)$ whenever $x>0$.

i.e. $e^x - (1+x) > 0$ so $e^x > 1+x$ when $x>0$.

Now take $h(x) = e^x - \left(1+x+\frac{x^2}{2}\right)$. I want to show $h(x)>0$ whenever $x>0$.

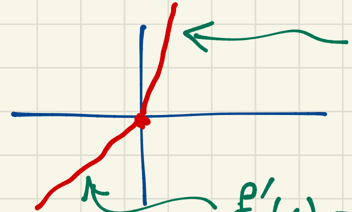
$h'(x) = e^x - (1+x) > 0$ for $x>0$. So h is increasing on $(0,\infty)$. So

$h(x) > h(0)$ whenever $x>0$. i.e. $e^x > 1+x+\frac{x^2}{2}$ whenever $x>0$.

$\lim_{x\to\infty} \frac{e^x}{x}$? This can be evaluated using l'Hopital's Rule which have not done yet. Or: For $x>0$, $\frac{e^x}{x} > \boxed{\frac{1+x+\frac{x^2}{2}}{x}} = \frac{1}{x} + 1 + \frac{x}{2} \xrightarrow{\text{as } x\to\infty} \infty$

$\lim_{x\to\infty} \frac{e^x}{x} = \infty$ by the Squeeze Theorem.

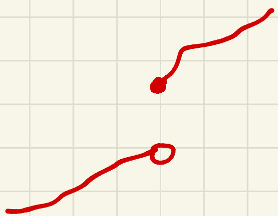
$$f(x) = \begin{cases} x, & x < 0; \\ 2x, & x \geq 0 \end{cases}$$



$f'(x) = 2$ for $x > 0$

$f'(x) = 1$ for $x < 0$

This function is increasing on $(-\infty, \infty)$. i.e. everywhere.



increasing

If $f' > 0$ on an interval then f is increasing on that interval.

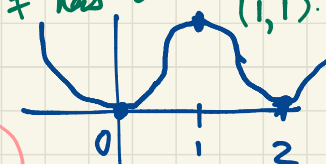
If $f' < 0$ " " " " " decreasing "

f has local max $(1,1)$.

p. 267 # 26. $f(x) = x^4 - 4x^3 + 4x^2 = x^2(x-2)^2$

$f'(x) = 4x^3 - 12x^2 + 8x = 4x(x^2 - 3x + 2) = 4x(x-2)(x-1)$

+ − +



Critical points: 0, 1, 2 where $f'(x) = 0$.

On $(-\infty, 0)$, $f' < 0$ so f is decreasing. on $(1,2)$, $f' < 0$ so f is decreasing.

on $(0, 1)$, $f' > 0$ so f is increasing on $(2, \infty)$, $f' > 0$ so f is increasing

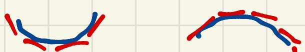
f has local and global minimum value 0 at 0 and at 2.

April 8

First derivative test: On intervals where $f'>0$, f is increasing.
" " " $f'<0$, f is decreasing.
At local extrema, f has a critical point ($f'=0$ or f' is undefined).
(Not conversely: critical points are not necessarily local extrema)

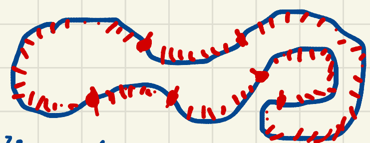
Second derivative test: if f'' exists on an interval

f



concave up	concave down
f' increasing	f' decreasing
$f''>0$	$f''<0$

f has an inflection point wherever the concavity changes



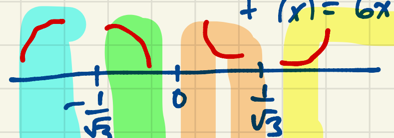
six inflection points on this curve

Put together the first and second derivative tests to work in sketching curves.

f		
concave up decreasing	$f'<0$	$f''>0$
concave up increasing	$f'>0$	$f''>0$
concave down decreasing	$f'<0$	$f''<0$
concave down increasing	$f'>0$	$f''<0$

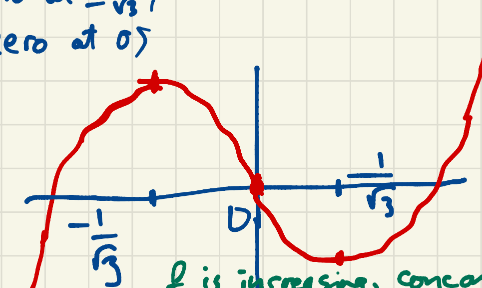
Example: $f(x)=x^3-x$
$f'(x)=3x^2-1$ (zero at $\pm\frac{1}{\sqrt3}$)
$f''(x)=6x$ (zero at 0)

f



f' + + + 0 − − − − 0 + + + +
f'' − − − − − 0 + + + + + + +

f is increasing, concave up on $(\frac{1}{\sqrt3},\infty)$
" " decreasing, " " " $(0,\frac{1}{\sqrt3})$
" concave down on $(-\frac{1}{\sqrt3},0)$



f is increasing, concave down on $(-\infty,-\frac{1}{\sqrt3})$.

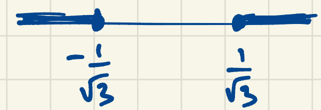
f has a local maximum point $\left(-\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$

f has a local minimum point $\left(\frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right)$

f has an inflection point $(0,0)$

Remark: f is actually increasing on $\left[\frac{1}{\sqrt{3}}, \infty\right)$. and on $\left(-\infty, -\frac{1}{\sqrt{3}}\right]$.

f is not increasing on $\left(-\infty, -\frac{1}{\sqrt{3}}\right] \cup \left[\frac{1}{\sqrt{3}}, \infty\right)$.

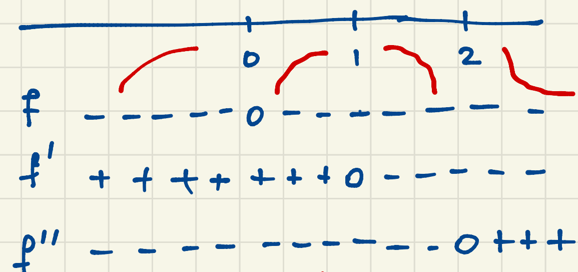


Eg. Find intervals where $f(x) = xe^{-x}$ is increasing, decreasing, concave up, and concave down and sketch the graph of f labelling local extrema and inflection points.

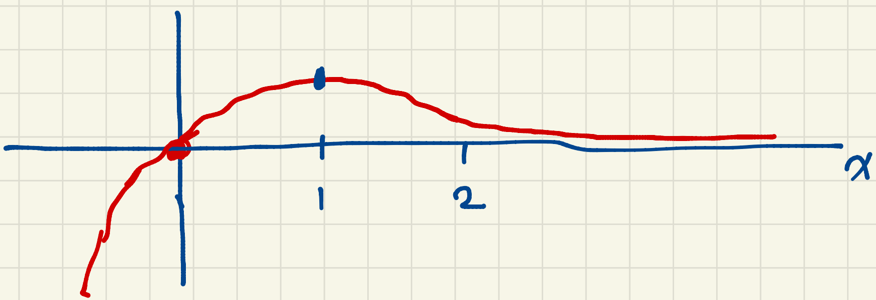
$f(x) = xe^{-x}$

$f'(x) = -xe^{-x} + e^{-x} = e^{-x}(1-x)$ (zero at 1)

$f''(x) = -e^{-x}(1-x) - e^{-x} = e^{-x}(x-2)$ (zero at 2)

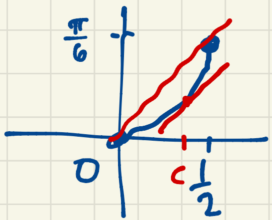


April 10



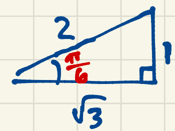
Another example from Sec 4.2 on the Mean Value Theorem:

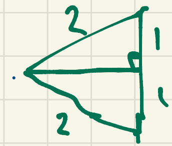
#29. $f(x) = \sin^{-1} x$, $\quad 0 \le x \le \frac{1}{2}$.



$\sin^{-1} \frac{1}{2} = ?$

$\sin \frac{\pi}{6} = \frac{1}{2}$





Average rate of change of f on $[0, \frac{1}{2}]$ is $\dfrac{f(\frac{1}{2}) - f(0)}{\frac{1}{2} - 0} = \dfrac{\frac{\pi}{6} - 0}{\frac{1}{2} - 0} = \dfrac{\pi}{3}$

$f'(x) = \dfrac{1}{\sqrt{1-x^2}}$

Find c, $0 < c < \frac{1}{2}$, such that $f'(c) = \frac{\pi}{3} = \dfrac{1}{\sqrt{1-c^2}}$. Solve for c.

$\dfrac{\pi^2}{9} = \dfrac{1}{1-c^2}$

$1 - c^2 = \dfrac{9}{\pi^2}$

$c^2 = 1 - \dfrac{9}{\pi^2}$

$c = \pm\sqrt{1 - \dfrac{9}{\pi^2}}$ (Only one solution in this case!)

$c = \sqrt{1 - \dfrac{9}{\pi^2}}$

Back to Sec 4.3

Note: $f(x) = x^2(2\ln x - 11)$

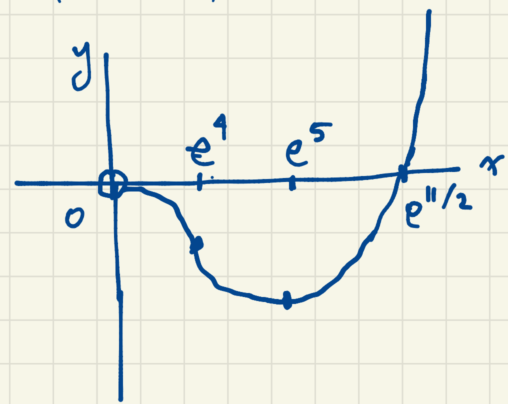
$f(x) > 0$ for $x > e^{11/2}$
$f(x) = 0$ for $x = e^{11/2}$
$f(x) < 0$ for $x < e^{11/2}$

\# 87. $f(x) = 2x^2 \ln x - 11x^2, \quad x > 0$

$f'(x) = 4x\ln x + 2x^2 \cdot \frac{1}{x} - 22x = 4x\ln x - 20x = 4x(\ln x - 5)$ zero when $x = e^5$

$f''(x) = 4\ln x + 4x \cdot \frac{1}{x} - 20 = 4\ln x - 16 = 4(\ln x - 4)$ zero when $x = e^4$

	0		e^4		e^5	
f		⌒		◡		◡
f'	0	$- - - -$	$- - - -$	$- -$	0	$+ + + +$
f''		$- - -$	0	$+ + +$	$+ + +$	$+$



f is increasing on (e^5, ∞), decreasing on $(0, e^5)$,
concave up on (e^4, ∞), concave down on $(0, e^4)$.
The inflection point is $(e^4, -3e^8)$. The local minimum point (e^5, e^{-10}) is also the absolute minimum point.

$f(e^5) = 2e^{10} \cdot 5 - 11 \cdot e^{10} = -e^{10}$

There is no absolute or local maximum. Since $\lim_{x\to\infty} f(x) = \infty$.

After we have covered l'Hopital's Rule we will explain $\lim_{x\to 0^+} f(x) = 0$.