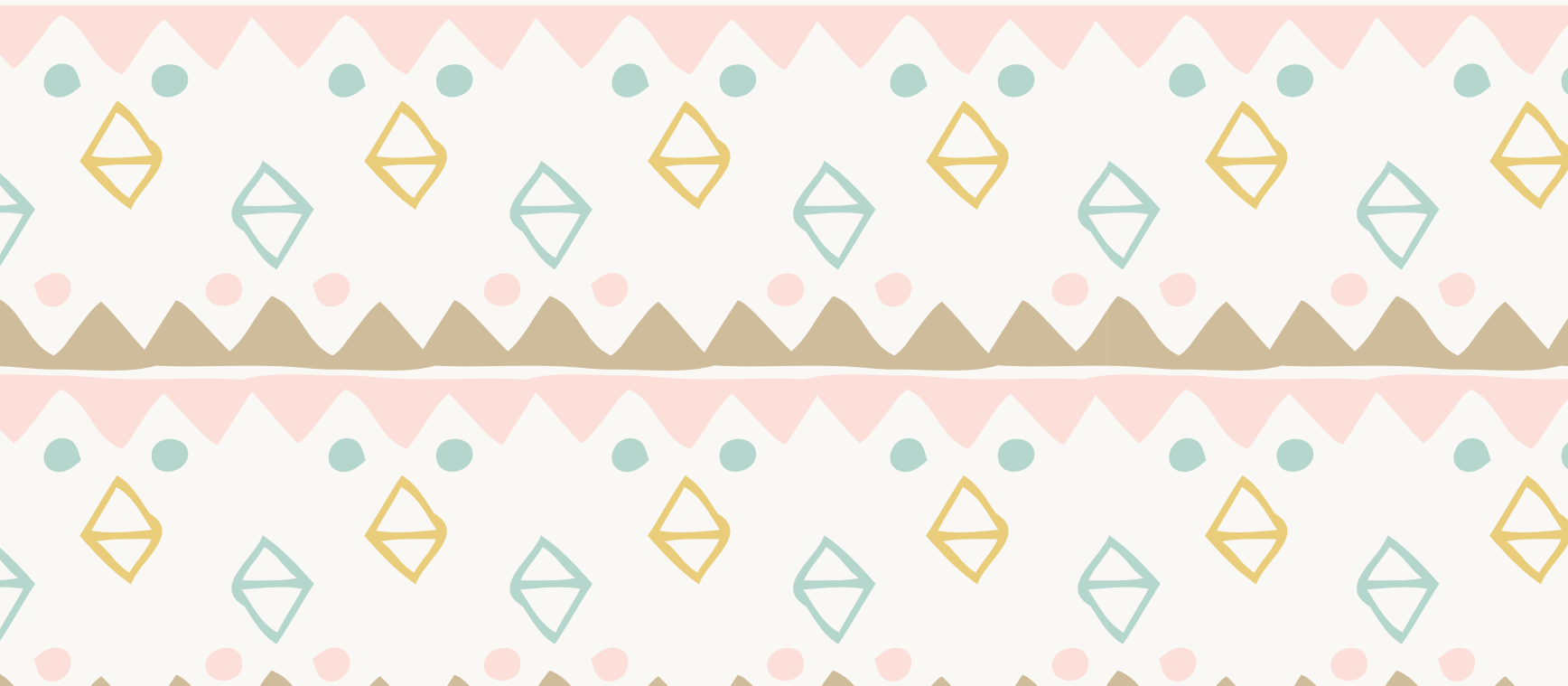


Math 2200-01 (Calculus I) Spring 2020

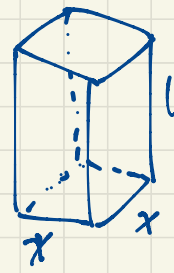
Book 3



Sec 4.5: Optimization

April 13

p.285 #19. of all boxes with a square base and a volume 8m^3 , which one has the minimum surface area?



volume $V = x^2 h = 8 \Rightarrow h = \frac{8}{x^2}$

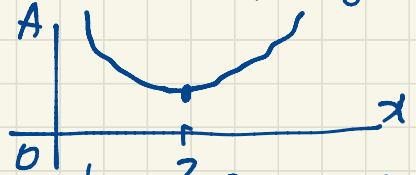
area $A = \underbrace{2x^2}_{\text{top and bottom}} + \underbrace{4xh}_{\text{sides}} = 2x^2 + 4x \cdot \frac{8}{x^2} = 2x^2 + \frac{32}{x}, x > 0$

The domain is $(0, \infty)$, an unbounded open interval.

$$\frac{dA}{dx} = 4x - \frac{32}{x^2} = \frac{1}{x^2} (x^3 - 8)$$

The critical point is at $x=2$.
 (Recall: critical points are where the derivative is zero or undefined. There are no points of the domain where $\frac{dA}{dx}$ is undefined, only one point where $\frac{dA}{dx} = 0$.)

For $0 < x < 2$, $\frac{dA}{dx} < 0$ so $A(x)$ is decreasing.
 For $x=2$, $\frac{dA}{dx} = 0$.
 For $x > 2$, $\frac{dA}{dx} > 0$ so $A(x)$ is increasing.

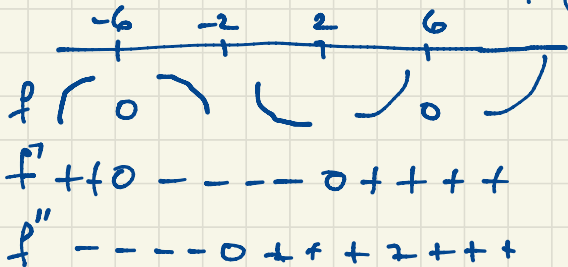


So the minimum surface area $A(2) = 12\text{m}^2$ occurs for a box of size $2\text{m} \times 2\text{m} \times 2\text{m}$.

Sec 4.4. p. 278 # 21. $f(x) = (x-6)(x+6)^2 = (x^2-36)(x+6) = x^3 + 6x^2 - 36x - 216$

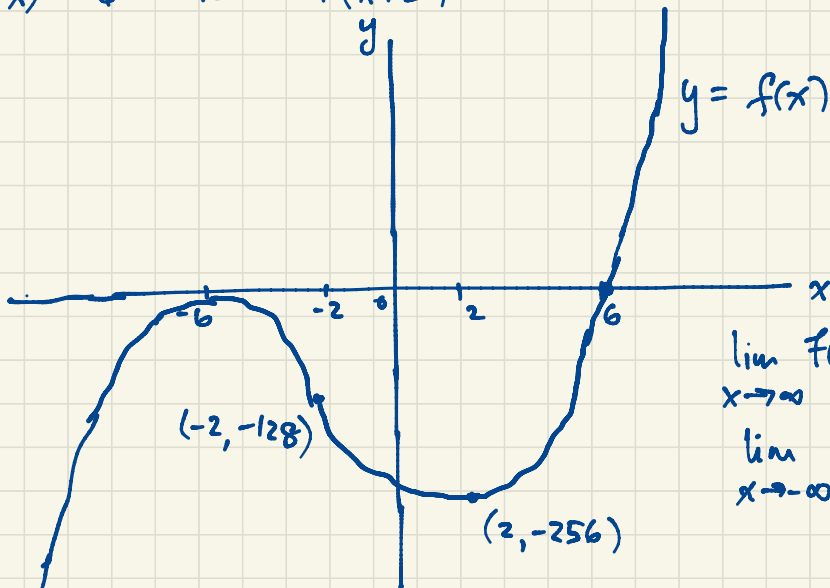
$$f'(x) = 3x^2 + 12x - 36 = 3(x^2 + 4x - 12) = 3(x+6)(x-2)$$

$$f''(x) = 6x + 12 = 6(x+2)$$



$$f(-2) = (-8)(4)^2 = -128$$

$$f(2) = (-4)(8^2) = -256$$



$$\lim_{x \rightarrow \infty} f(x) = \infty$$

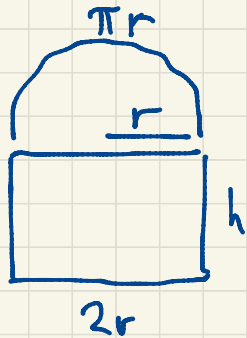
$$\lim_{x \rightarrow -\infty} f(x) = -\infty$$

April 14

f is increasing on $(-\infty, -6)$ and on $(2, \infty)$,
 decreasing on $(-6, 2)$,
 concave down on $(-\infty, -2)$,
 concave up on $(-2, \infty)$.

f has an inflection point $(-2, -128)$,
 a local minimum point $(2, -256)$,
 a local maximum point $(-6, 0)$,
 no absolute extrema or asymptotes.

Sec 4.5 p. 287 #41. Let r be the radius of the semicircular window pane.



The perimeter is $P = \pi r + 2r + 2h = 20$

$$(\pi + 2)r + 2h = 20$$

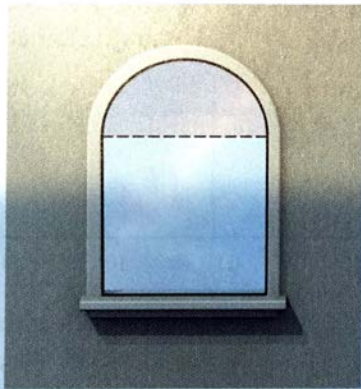
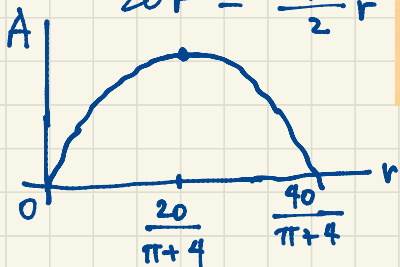
$$2h = 20 - \pi r - 2r$$

$$h = 10 - \frac{\pi + 2}{2}r$$

41. A window consists of rectangular pane of glass surmounted by a semicircular pane of glass (see figure). If the perimeter of the window is 20 feet, determine the dimensions of the window that maximize the area of the window.

$$\frac{dA}{dr} = 20 - (\pi + 4)r$$

$$\begin{aligned} A &= \frac{\pi}{2}r^2 + 2rh \\ &= \frac{\pi}{2}r^2 + 2r\left(10 - \frac{\pi + 2}{2}r\right) \\ &= 20r + \left(\frac{\pi}{2} - (\pi + 2)\right)r^2 \\ &= 20r - \frac{\pi + 4}{2}r^2 \end{aligned}$$



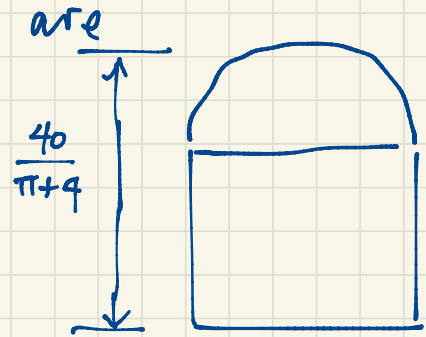
The critical point is at $r = \frac{20}{\pi + 4}$.
When $0 < r < \frac{20}{\pi + 4}$, $\frac{dA}{dr} > 0$ so A is increasing.

When $\frac{20}{\pi + 4} < r < \frac{40}{\pi + 4}$, $\frac{dA}{dr} < 0$ so A is decreasing.

$$A = \left(20 - \frac{\pi + 4}{2}r\right)r$$

So the maximum area occurs when $r = \frac{20}{\pi + 4}$. Alternatively since $A \geq 0$ requires r to be in $\left[0, \frac{40}{\pi + 4}\right]$, we need only check A at endpoints and the critical point.

The dimensions of the window that maximizes the area



$$r = \frac{20}{\pi+4}$$

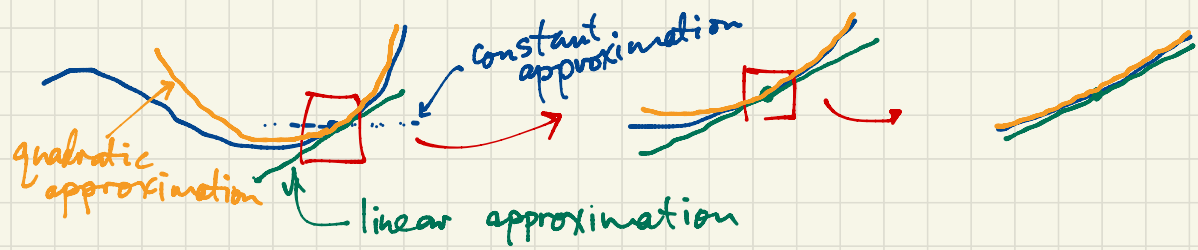
$$h = 10 - \frac{\pi+2}{2}r = 10 - \frac{\pi+2}{2} \cdot \frac{20}{\pi+4}$$

$$= 10 - \frac{10(\pi+2)}{\pi+4}$$

$$= \frac{10(\pi+4) - 10(\pi+2)}{\pi+4}$$

$$2r = \frac{40}{\pi+4}$$

Sec 4.6. Linearization and Differentials $= \frac{20}{\pi+4}$



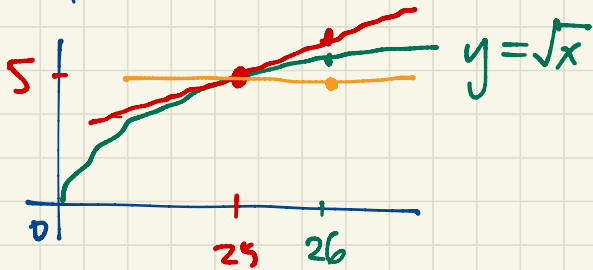
April 15

For x close to a , $f(x) \approx L(x) = f'(a)(x-a) + f(a) = \boxed{}x + \boxed{}$

approximately equal to $\underbrace{\hspace{10em}}_{\text{linearization of } f \text{ at } (a, f(a))}$

$f'(a)$ $f(a) - af'(a)$

Example: Use the linearization of \sqrt{x} at $(25, 5)$ to approximate $\sqrt{26}$.



$$f(x) = \sqrt{x}, \quad f(25) = 5$$

$$f'(x) = \frac{1}{2\sqrt{x}}, \quad f'(25) = \frac{1}{2\sqrt{25}} = \frac{1}{10}$$

The linearization of \sqrt{x} at $(25, 5)$ is

$$\begin{aligned} L(x) &= f'(25)(x-25) + f(25) \\ &= \frac{1}{10}(x-25) + 5. \end{aligned}$$

If $x \approx 25$ then $\sqrt{x} \approx \frac{1}{10}(x-25) + 5$.

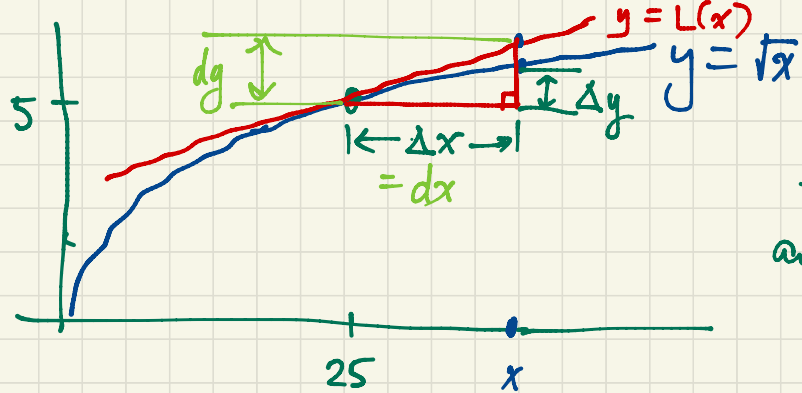
$$\text{Eq. } \sqrt{26} \approx \frac{1}{10}(26-25) + 5 = 5.1$$

$$\text{Check: } \sqrt{26} \approx 5.099019514$$

correct to 3 significant digits.

Coarser approximation: $\sqrt{26} \approx 5$. (correct to one significant digit)

Constant approximation $\sqrt{x} \approx 5$



If we move from $(25, 5)$ to $(x, f(x))$ on the graph, our actual function $f(x) = \sqrt{x}$ changes by an exact amount

$$\Delta g = f(x) - f(25).$$

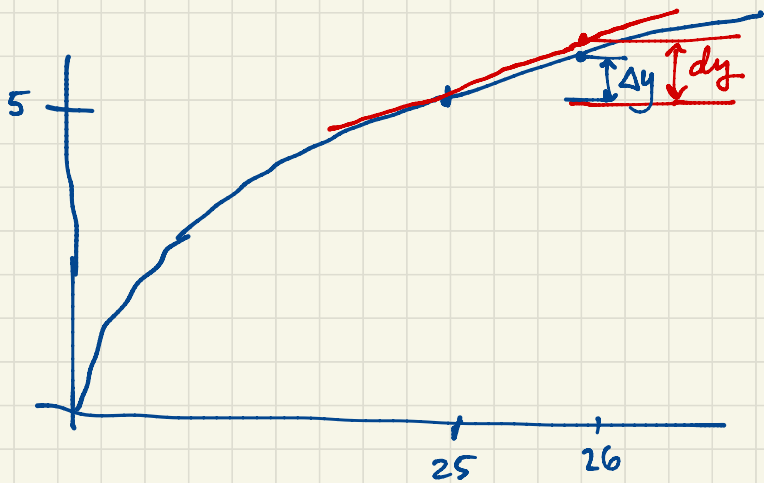
The secant line from $(25, 5)$ to $(x, f(x))$ has slope $\frac{\Delta g}{\Delta x} = \frac{f(x) - f(25)}{x - 25}$.

On the tangent line,

$$\frac{\Delta L(x)}{\Delta x} = \frac{L(x) - L(25)}{x - 25} = f'(25) = \frac{dy}{dx}$$

Until now we have written $\frac{dy}{dx}$ as an indivisible symbol. Now we are interpreting dx and dy as changes in x and y . They are changes on the tangent line (just like Δx and Δy are corresponding changes on the actual curve of f). dx and dy are differentials.

We'll interpret $\sqrt{26} \approx 5.1$ in this new language:



$$f'(25) = \frac{1}{10} = \frac{dy}{dx} \Rightarrow dy = \frac{1}{10} dx = \frac{1}{10} \times 1 = 0.1$$

$$dx = \Delta x = 26 - 25 = 1$$

As we move from $x=25$ to $x=26$, the corresponding change in y is

$$\Delta y \approx dy = 0.1.$$

$$\text{So } \sqrt{26} \approx 5.1$$

This is a quick and easy interpretation for differentials. We will be using differentials throughout Calculus.

Integrals $\int_a^b f(x) dx$

$$x \rightarrow u \rightarrow y$$

$$\frac{dy}{dx} = \frac{dy}{\cancel{du}} \cdot \frac{\cancel{du}}{dx}$$

If $y = \sin x$, find $\frac{dy}{dx}$ and dy .

$$\frac{dy}{dx} = \cos x \quad \text{so} \quad dy = (\cos x) dx$$

$$\text{so } \Delta y \approx dy = \cos x dx$$

Sec 1.7 l'Hôpital's Rule

The limit $\lim_{x \rightarrow \infty} \frac{x^3 + 2x^2 - x - 1}{2x^3 - 5x} = \lim_{x \rightarrow \infty} \frac{1 + \frac{2}{x} - \frac{1}{x^2} - \frac{1}{x^3}}{2 - \frac{5}{x^2}} = \frac{1}{2}$.

"indeterminate form $\frac{\infty}{\infty}$ "

determinate form

The limit $\lim_{x \rightarrow \infty} \frac{1}{3x^2 + 5} = 0$.

"determinate form $\frac{1}{\infty}$ "

Do not confuse l'Hôpital's Rule with the Quotient Rule!

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

indeterminate

form $\frac{0}{0}$

l'Hôpital's Rule For limits of indeterminate form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, $\lim_x \frac{f(x)}{g(x)} = \lim_x \frac{f'(x)}{g'(x)}$

assuming the second limit exists.

eg. $\lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \frac{\cos x}{1} = 1$.

eg. $\lim_{x \rightarrow \infty} \frac{x^3 + 2x^2 - x - 1}{2x^3 - 5x} = \lim_{x \rightarrow \infty} \frac{3x^2 + 4x - 1}{6x^2 - 5} = \lim_{x \rightarrow \infty} \frac{6x + 4}{12x} = \lim_{x \rightarrow \infty} \frac{6}{12} = \frac{1}{2}$

$$\text{eg. } \lim_{x \rightarrow \infty} x e^{-x} = \lim_{x \rightarrow \infty} \frac{x}{e^x} = \lim_{x \rightarrow \infty} \frac{1}{e^x} = 0$$

indeterminate form $\infty \cdot 0$ indeterminate form $\frac{\infty}{\infty}$

$$\text{eg. } \lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0^+} (-x) = 0.$$

indeterminate form $0 \cdot (-\infty)$ indeterminate form $\frac{-\infty}{\infty}$

↙ Don't keep using l'Hôpital's Rule beyond this point; that approach never reaches a conclusion.

$$\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{x}{\frac{1}{\ln x}} = \lim_{x \rightarrow 0^+} \frac{1}{-\frac{1}{(\ln x)^2} \cdot \frac{1}{x}} = \lim_{x \rightarrow 0^+} (-x (\ln x)^2)$$

indeterminate form $\frac{0}{0}$

(a-b)(a+b) = a^2 - b^2

this is not an improvement!

$$\lim_{x \rightarrow \infty} \left(\sqrt{x^2 + 10x} - x \right) \cdot \frac{\sqrt{x^2 + 10x} + x}{\sqrt{x^2 + 10x} + x} = \lim_{x \rightarrow \infty} \frac{x^2 + 10x - x^2}{\sqrt{x^2 + 10x} + x} = \lim_{x \rightarrow \infty} \frac{10x}{\sqrt{x^2 + 10x} + x}$$

indeterminate form $\infty - \infty$

$$= \lim_{x \rightarrow \infty} \frac{10}{\sqrt{1 + \frac{10}{x}} + 1} = \frac{10}{\sqrt{1+0} + 1} = 5.$$

Note:

$$\sqrt{x^2 + 10x} = \sqrt{x^2 \left(1 + \frac{10}{x}\right)} = x \sqrt{1 + \frac{10}{x}}$$

$$\sqrt{x^2} = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

(April 20)

Another l'Hôpital's Rule example for the indeterminate form 1^∞ .

$$\lim_{x \rightarrow \infty} \left(1 + \frac{a}{x}\right)^x \text{ where } a \text{ is constant.}$$

In order to use l'Hôpital's Rule, we need to rewrite this as a quotient.

$$b^c = (e^{\ln b})^c = e^{c \ln b}$$

$$\lim_{x \rightarrow \infty} \left(1 + \frac{a}{x}\right)^x = \lim_{x \rightarrow \infty} e^{x \ln \left(1 + \frac{a}{x}\right)} = \lim_{u \rightarrow 1} e^{\frac{a}{u-1} \ln u} = e^{a \cdot 1} = e^a.$$

$$u = 1 + \frac{a}{x} \rightarrow 1.$$

$$u-1 = \frac{a}{x}$$

$$x = \frac{a}{u-1}$$

$$\lim_{u \rightarrow 1} \frac{\ln u}{u-1} = \lim_{u \rightarrow 1} \frac{1/u}{1} = 1.$$

(indeterminate form $\frac{0}{0}$)

$$\text{If } f(x) = e^{ax} \text{ then } \lim_{u \rightarrow 1} f\left(\frac{\ln u}{u-1}\right) = f\left(\lim_{u \rightarrow 1} \frac{\ln u}{u-1}\right) = f(1) = e^{a \cdot 1} = e^a.$$

Does $\lim_u f(g(u)) = f(\lim_u g(u))$? Can you move the limit inside?

We can do this when f is continuous and g is continuous.

$$c = \lim_u g(u), \text{ write } t = g(u) \rightarrow c. \quad \lim_{t \rightarrow c} f(t) = f(c).$$

Some books define $e^r = \lim_{x \rightarrow \infty} \left(1 + \frac{r}{x}\right)^x$.

This limit comes from compound interest.

If you deposit a principal amount A in the bank at nominal interest rate r (eg. 5% interest per annum would give $r = 0.05$). If interest is compounded annually then after one year you have earned rA interest. The total balance in the bank after a year would be $A + rA = (1+r)A$.

If interest is compounded semiannually (every 6 months) then after 6 months you have $(1 + \frac{r}{2})A$ as balance after 6 months; then at the end of the year you have $(1 + \frac{r}{2}) \cdot (1 + \frac{r}{2})A = (1 + \frac{r}{2})^2 A = (1 + r + \frac{r^2}{4})A$.

If interest is compounded n times per year then every $\frac{1}{n}$ year your balance is multiplied by $1 + \frac{r}{n}$. After one year your balance is $(1 + \frac{r}{n})^n A$.

For continually compounded interest we let $n \rightarrow \infty$.

$\lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^n A = e^r A$. Eg. 5% interest compounded continuously results in

balance $e^{0.05} A \approx 1.05127 A$ (effectively 5.127% interest rate per annum).

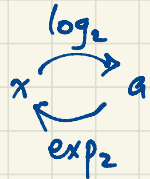
p. 31 # 69. $\lim_{x \rightarrow \infty} \frac{\log_2 x}{\log_3 x} = \lim_{x \rightarrow \infty} \frac{(\ln x) / \ln 2}{(\ln x) / \ln 3} = \frac{\ln 3}{\ln 2} = \log_2 3$

$\log_2 x = \frac{\ln x}{\ln 2}$ why?

$a = \log_2 x \iff 2^a = x \iff \ln(2^a) = \ln x$

$\iff a \ln 2 = \ln x$

$\iff a = \frac{\ln x}{\ln 2}$



April 21

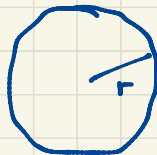
Sec 4.6 Linearization and Differentials

$y = f(x)$

$\frac{dy}{dx} = f'(x)$

$\Delta y \approx dy = f'(x) dx = f'(x) \Delta x$
small change in y
due to change
in x .

Formulas for area and circumference of a circle

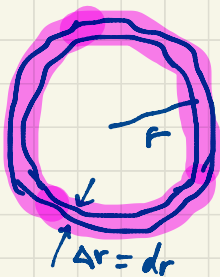


$A = \pi r^2$

$C = 2\pi r$

From the formula $A = \pi r^2$ you can deduce the formula $C = 2\pi r$ using linearization.

Starting with a circle of radius r , area $A = \pi r^2$, enlarge the circle by expanding the radius by a small amount Δr .



$$\Delta A \approx C \Delta r \approx C \Delta r$$

$$\Delta A \approx dA = 2\pi r dr = 2\pi r \Delta r$$

$$A = \pi r^2$$

$$\frac{dA}{dr} = 2\pi r$$

$$dA = 2\pi r dr$$

$$\text{So } C = 2\pi r.$$

For a sphere of radius r ,

$$V = \frac{4}{3}\pi r^3. \quad \text{What is the surface area?}$$

Imagine painting the surface of a ball of radius r . How much paint is required to paint the surface? If I add a layer of paint of thickness Δr then the new volume changes by

$$\Delta V \approx A \Delta r$$

$$\Delta V \approx dV = 4\pi r^2 dr = 4\pi r^2 \Delta r$$

$$\text{So } A = 4\pi r^2.$$

Sec 4.6 p. 299 #28. $f(x) = \frac{x}{x+1}$. Approximate $f(1.1)$ using the linearization at $x=1$.

$f(1) = \frac{1}{2} = 0.5$. This value is the base for our linearization.

$$L(x) = f(a) + f'(a)(x-a), \quad a=1$$
$$= \frac{1}{2} + \frac{1}{4}(x-1)$$

$$f(1.1) \approx L(1.1) = \frac{1}{2} + \frac{1}{4}(0.1) = 0.5 + 0.025 = 0.525$$

$$\frac{dy}{dx} = f'(x) = \frac{(x+1) \cdot 1 - x \cdot 1}{(x+1)^2}$$

$$= \frac{1}{(x+1)^2}$$

$$dy = \frac{dx}{(x+1)^2}$$

Alternatively using differentials:

x goes from 1 to 1.1, a small change of $\Delta x = dx = 0.1$. What is the corresponding change in y ?

$$\Delta y \approx dy = \frac{dx}{(x+1)^2}$$

$$\Delta y \approx \frac{\Delta x}{(1+1)^2} = \frac{\Delta x}{4} = \frac{0.1}{4} = 0.025$$

$$y \approx 0.5 + \Delta y = 0.525.$$

Actually $0.0023 = 0.23\%$
"per cent" means "divided by 100"

$$f(1.1) = \frac{1.1}{2.1} = 0.52380952\dots$$

The error is $0.525 - f(1.1) = 0.00119\dots$

The relative error is $\frac{0.00119}{0.5238} = 0.0023\dots$

(an over estimate since this difference is positive).

The percentage error is $0.0023 \times 100\% = 0.23\%$
(about a quarter of a percent error).

Sec 4.9 Antiderivatives

$$\frac{d}{dx} (7x^2+2x+1)^3 = 3(7x^2+2x+1)^2(14x+2).$$

Reverse: ~~The~~ ^{An} antiderivative of $3(7x^2+2x+1)^2(14x+2)$ is $(7x^2+2x+1)^3$.

The general answer for antiderivative: $(7x^2+2x+1)^3 + \text{constant}$.

$$\frac{d}{dx} [(7x^2+2x+1)^3 + 11] = 3(7x^2+2x+1)^2(14x+2).$$

We write $\int 3(7x^2+2x+1)^2(14x+2) dx = (7x^2+2x+1)^3 + C$ where C is an arbitrary

The general antiderivative of $3(7x^2+2x+1)^2(14x+2)$ with constant respect to x is $(7x^2+2x+1)^3 + \text{constant}$.