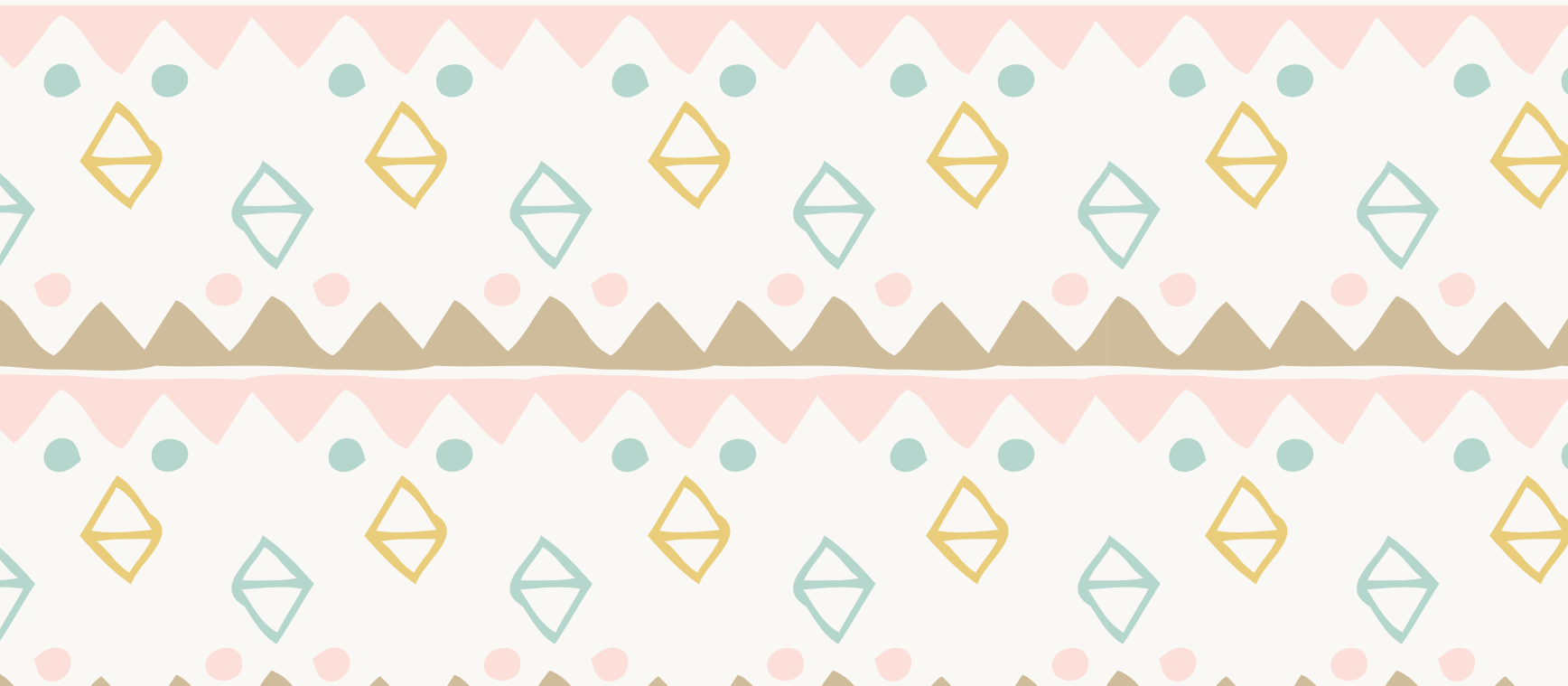


# Math 2200-01 (Calculus I) Spring 2020

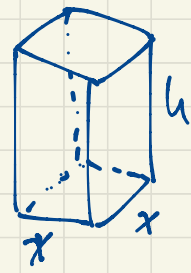
Book 3



# Sec 4.5: Optimization

April 13

p.285 #19. of all boxes with a square base and a volume  $8\text{m}^3$ , which one has the minimum surface area?



volume  $V = x^2 h = 8 \Rightarrow h = \frac{8}{x^2}$

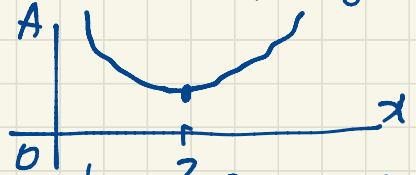
area  $A = \underbrace{2x^2}_{\text{top and bottom}} + \underbrace{4xh}_{\text{sides}} = 2x^2 + 4x \cdot \frac{8}{x^2} = 2x^2 + \frac{32}{x}, x > 0$

The domain is  $(0, \infty)$ , an unbounded open interval.

$$\frac{dA}{dx} = 4x - \frac{32}{x^2} = \frac{1}{x^2} (x^3 - 8)$$

The critical point is at  $x=2$ .  
 (Recall: critical points are where the derivative is zero or undefined. There are no points of the domain where  $\frac{dA}{dx}$  is undefined, only one point where  $\frac{dA}{dx} = 0$ .)

For  $0 < x < 2$ ,  $\frac{dA}{dx} < 0$  so  $A(x)$  is decreasing.  
 For  $x=2$ ,  $\frac{dA}{dx} = 0$ .  
 For  $x > 2$ ,  $\frac{dA}{dx} > 0$  so  $A(x)$  is increasing.

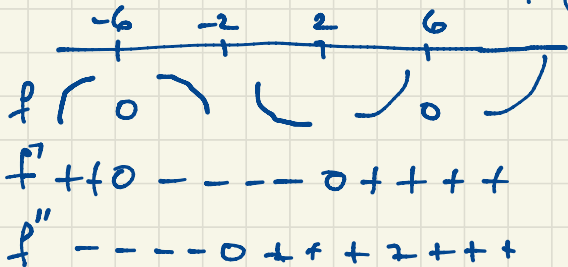


So the minimum surface area  $A(2) = 12\text{m}^2$  occurs for a box of size  $2\text{m} \times 2\text{m} \times 2\text{m}$ .

Sec 4.4. p. 278 # 21.  $f(x) = (x-6)(x+6)^2 = (x^2-36)(x+6) = x^3 + 6x^2 - 36x - 216$

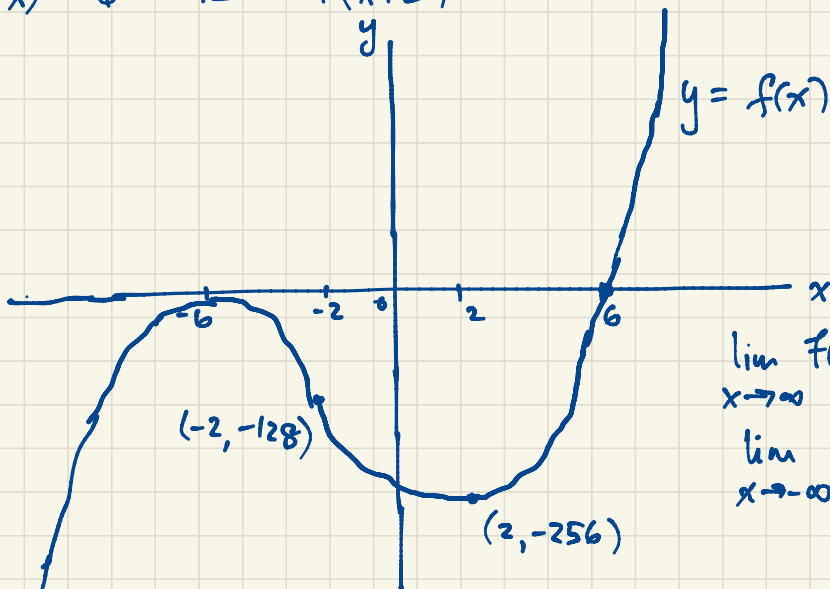
$$f'(x) = 3x^2 + 12x - 36 = 3(x^2 + 4x - 12) = 3(x+6)(x-2)$$

$$f''(x) = 6x + 12 = 6(x+2)$$



$$f(-2) = (-8)(4)^2 = -128$$

$$f(2) = (-4)(8^2) = -256$$



$$\lim_{x \rightarrow \infty} f(x) = \infty$$

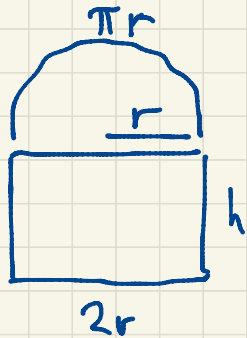
$$\lim_{x \rightarrow -\infty} f(x) = -\infty$$

April 14

$f$  is increasing on  $(-\infty, -6)$  and on  $(2, \infty)$ ,  
 decreasing on  $(-6, 2)$ ,  
 concave down on  $(-\infty, -2)$ ,  
 concave up on  $(-2, \infty)$ .

$f$  has an inflection point  $(-2, -128)$ ,  
 a local minimum point  $(2, -256)$ ,  
 a local maximum point  $(-6, 0)$ ,  
 no absolute extrema or asymptotes.

Sec 4.5 p. 287 #41. Let  $r$  be the radius of the semicircular window pane.



The perimeter is  $P = \pi r + 2r + 2h = 20$

$$(\pi + 2)r + 2h = 20$$

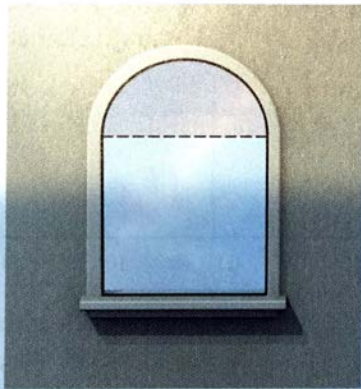
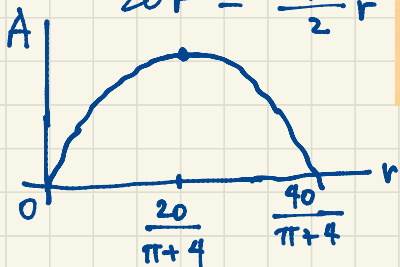
$$2h = 20 - \pi r - 2r$$

$$h = 10 - \frac{\pi + 2}{2}r$$

41. A window consists of rectangular pane of glass surmounted by a semicircular pane of glass (see figure). If the perimeter of the window is 20 feet, determine the dimensions of the window that maximize the area of the window.

$$\frac{dA}{dr} = 20 - (\pi + 4)r$$

$$\begin{aligned} A &= \frac{\pi}{2}r^2 + 2rh \\ &= \frac{\pi}{2}r^2 + 2r\left(10 - \frac{\pi + 2}{2}r\right) \\ &= 20r + \left(\frac{\pi}{2} - (\pi + 2)\right)r^2 \\ &= 20r - \frac{\pi + 4}{2}r^2 \end{aligned}$$



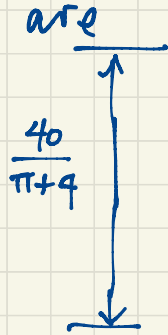
$$A = \left(20 - \frac{\pi + 4}{2}r\right)r$$

The critical point is at  $r = \frac{20}{\pi + 4}$ .  
When  $0 < r < \frac{20}{\pi + 4}$ ,  $\frac{dA}{dr} > 0$  so  $A$  is increasing.

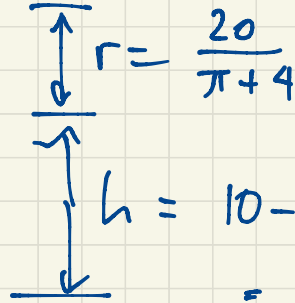
When  $\frac{20}{\pi + 4} < r < \frac{40}{\pi + 4}$ ,  $\frac{dA}{dr} < 0$  so  $A$  is decreasing.

So the maximum area occurs when  $r = \frac{20}{\pi + 4}$ . Alternatively since  $A \geq 0$  requires  $r$  to be in  $\left[0, \frac{40}{\pi + 4}\right]$ , we need only check  $A$  at endpoints and the critical point.

The dimensions of the window that maximizes the area



$$2r = \frac{40}{\pi+4}$$

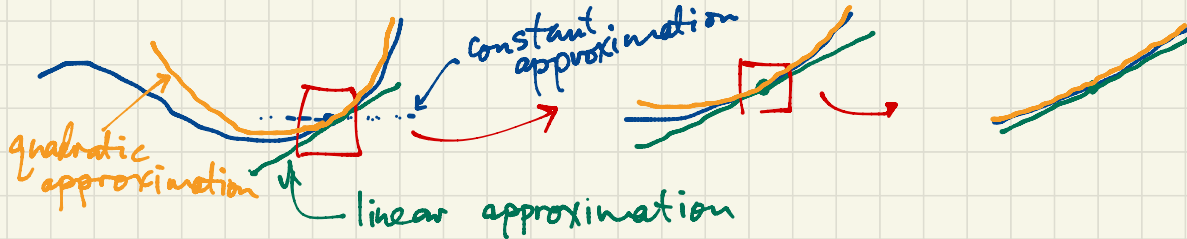


$$h = 10 - \frac{\pi+2}{2}r = 10 - \frac{\pi+2}{2} \cdot \frac{20}{\pi+4}$$

$$= 10 - \frac{10(\pi+2)}{\pi+4}$$

$$= \frac{10(\pi+4) - 10(\pi+2)}{\pi+4}$$

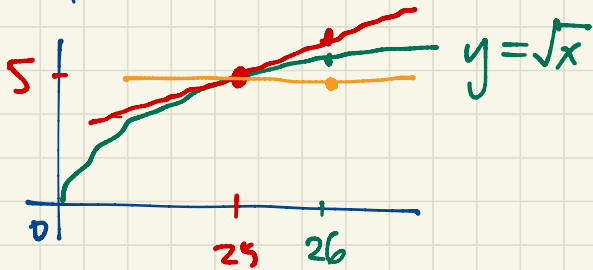
## Sec 4.6. Linearization and Differentials $= \frac{20}{\pi+4}$



April 15

For  $x$  close to  $a$ ,  $f(x) \approx L(x) = f'(a)(x-a) + f(a) = \boxed{\phantom{00}}x + \boxed{\phantom{00}}$   
 approximately equal to linearization of  $f$  at  $(a, f(a))$   
 $f'(a)$   $f(a) - af'(a)$

Example: Use the linearization of  $\sqrt{x}$  at  $(25, 5)$  to approximate  $\sqrt{26}$ .



$$f(x) = \sqrt{x}, \quad f(25) = 5$$

$$f'(x) = \frac{1}{2\sqrt{x}}, \quad f'(25) = \frac{1}{2\sqrt{25}} = \frac{1}{10}$$

The linearization of  $\sqrt{x}$  at  $(25, 5)$  is

$$\begin{aligned} L(x) &= f'(25)(x-25) + f(25) \\ &= \frac{1}{10}(x-25) + 5. \end{aligned}$$

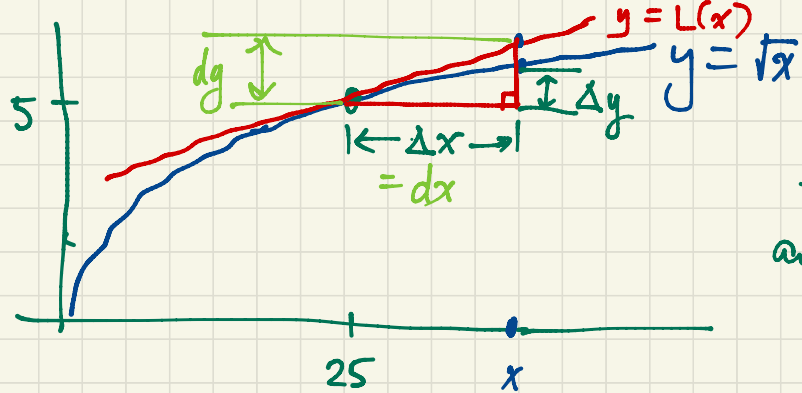
If  $x \approx 25$  then  $\sqrt{x} \approx \frac{1}{10}(x-25) + 5$ .

$$\text{Eq. } \sqrt{26} \approx \frac{1}{10}(26-25) + 5 = 5.1$$

$$\text{Check: } \sqrt{26} \approx 5.099019514$$

correct to 3 significant digits.

Coarser approximation:  $\sqrt{26} \approx 5$ . (correct to one significant digit)  
 Constant approximation  $\sqrt{x} \approx 5$



If we move from  $(25, 5)$  to  $(x, f(x))$  on the graph, our actual function  $f(x) = \sqrt{x}$  changes by an exact amount

$$\Delta g = f(x) - f(25).$$

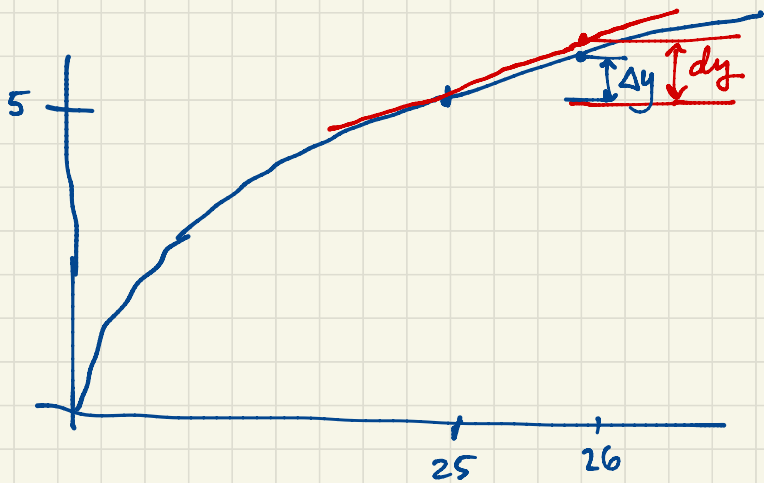
The secant line from  $(25, 5)$  to  $(x, f(x))$  has slope  $\frac{\Delta g}{\Delta x} = \frac{f(x) - f(25)}{x - 25}$ .

On the tangent line,

$$\frac{\Delta L(x)}{\Delta x} = \frac{L(x) - L(25)}{x - 25} = f'(25) = \frac{dy}{dx}$$

Until now we have written  $\frac{dy}{dx}$  as an indivisible symbol. Now we are interpreting  $dx$  and  $dy$  as changes in  $x$  and  $y$ . They are changes on the tangent line (just like  $\Delta x$  and  $\Delta y$  are corresponding changes on the actual curve of  $f$ ).  $dx$  and  $dy$  are differentials.

We'll interpret  $\sqrt{26} \approx 5.1$  in this new language:



$$f'(25) = \frac{1}{10} = \frac{dy}{dx} \Rightarrow dy = \frac{1}{10} dx = \frac{1}{10} \times 1 = 0.1$$

$$dx = \Delta x = 26 - 25 = 1$$

As we move from  $x=25$  to  $x=26$ ,  
the corresponding change in  $y$  is

$$\Delta y \approx dy = 0.1.$$

$$\text{So } \sqrt{26} \approx 5.1$$

This is a quick and easy interpretation for differentials. We will be using differentials throughout Calculus.

Integrals  $\int_a^b f(x) dx$

$$x \rightarrow u \rightarrow y$$

$$\frac{dy}{dx} = \frac{dy}{\cancel{du}} \cdot \frac{\cancel{du}}{dx}$$

If  $y = \sin x$ , find  $\frac{dy}{dx}$  and  $dy$ .

$$\frac{dy}{dx} = \cos x \quad \text{so} \quad dy = (\cos x) dx$$

$$\text{so } \Delta y \approx dy = \cos x dx$$

## Sec 1.7 l'Hôpital's Rule

The limit  $\lim_{x \rightarrow \infty} \frac{x^3 + 2x^2 - x - 1}{2x^3 - 5x} = \lim_{x \rightarrow \infty} \frac{1 + \frac{2}{x} - \frac{1}{x^2} - \frac{1}{x^3}}{2 - \frac{5}{x^2}} = \frac{1}{2}$ .

"indeterminate form  $\frac{\infty}{\infty}$ "

determinate form

The limit  $\lim_{x \rightarrow \infty} \frac{1}{3x^2 + 5} = 0$ .

"determinate form  $\frac{1}{\infty}$ "

Do not confuse l'Hôpital's Rule with the Quotient Rule!

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

indeterminate

form  $\frac{0}{0}$

l'Hôpital's Rule For limits of indeterminate form  $\frac{0}{0}$  or  $\frac{\pm\infty}{\pm\infty}$ ,  $\lim_x \frac{f(x)}{g(x)} = \lim_x \frac{f'(x)}{g'(x)}$

assuming the second limit exists.

eg.  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \frac{\cos x}{1} = 1$ .

eg.  $\lim_{x \rightarrow \infty} \frac{x^3 + 2x^2 - x - 1}{2x^3 - 5x} = \lim_{x \rightarrow \infty} \frac{3x^2 + 4x - 1}{6x^2 - 5} = \lim_{x \rightarrow \infty} \frac{6x + 4}{12x} = \lim_{x \rightarrow \infty} \frac{6}{12} = \frac{1}{2}$

eg.  $\lim_{x \rightarrow \infty} x e^{-x} = \lim_{x \rightarrow \infty} \frac{x}{e^x} = \lim_{x \rightarrow \infty} \frac{1}{e^x} = 0$

indeterminate form  $\infty \cdot 0$       indeterminate form  $\frac{\infty}{\infty}$

eg.  $\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0^+} (-x) = 0.$

indeterminate form  $0 \cdot (-\infty)$       indeterminate form  $\frac{-\infty}{\infty}$

Don't keep using l'Hôpital's Rule beyond this point; that approach never reaches a conclusion.

$\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{x}{\frac{1}{\ln x}} = \lim_{x \rightarrow 0^+} \frac{1}{-\frac{1}{(\ln x)^2} \cdot \frac{1}{x}} = \lim_{x \rightarrow 0^+} (-x (\ln x)^2)$       this is not an improvement!

indeterminate form  $\frac{0}{0}$

$$(a-b)(a+b) = a^2 - b^2$$

$\lim_{x \rightarrow \infty} (\sqrt{x^2 + 10x} - x) \cdot \frac{\sqrt{x^2 + 10x} + x}{\sqrt{x^2 + 10x} + x} = \lim_{x \rightarrow \infty} \frac{x^2 + 10x - x^2}{\sqrt{x^2 + 10x} + x} = \lim_{x \rightarrow \infty} \frac{10x}{\sqrt{x^2 + 10x} + x}$

indeterminate form  $\infty - \infty$

$$= \lim_{x \rightarrow \infty} \frac{10}{\sqrt{1 + \frac{10}{x}} + 1} = \frac{10}{\sqrt{1+0} + 1} = 5.$$

Note:

$$\sqrt{x^2 + 10x} = \sqrt{x^2 \left(1 + \frac{10}{x}\right)} = x \sqrt{1 + \frac{10}{x}}$$

$$\sqrt{x^2} = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

April 20

Another l'Hôpital's Rule example for the indeterminate form  $1^\infty$ .

$$\lim_{x \rightarrow \infty} \left(1 + \frac{a}{x}\right)^x \text{ where } a \text{ is constant.}$$

In order to use l'Hôpital's Rule, we need to rewrite this as a quotient.

$$b^c = (e^{\ln b})^c = e^{c \ln b}$$

$$\lim_{x \rightarrow \infty} \left(1 + \frac{a}{x}\right)^x = \lim_{x \rightarrow \infty} e^{x \ln \left(1 + \frac{a}{x}\right)} = \lim_{u \rightarrow 1} e^{\frac{a}{u-1} \ln u} = e^{a \cdot 1} = e^a.$$

$$u = 1 + \frac{a}{x} \rightarrow 1.$$

$$\lim_{u \rightarrow 1} \frac{\ln u}{u-1} = \lim_{u \rightarrow 1} \frac{1/u}{1} = 1.$$

$$u-1 = \frac{a}{x}$$

$$x = \frac{a}{u-1}$$

(indeterminate form  $\frac{0}{0}$ )

$$\text{If } f(x) = e^{ax} \text{ then } \lim_{u \rightarrow 1} f\left(\frac{\ln u}{u-1}\right) = f\left(\lim_{u \rightarrow 1} \frac{\ln u}{u-1}\right) = f(1) = e^{a \cdot 1} = e^a.$$

Does  $\lim_u f(g(u)) = f(\lim_u g(u))$ ? Can you move the limit inside?

We can do this when  $f$  is continuous and  $g$  is continuous.

$$c = \lim_u g(u), \text{ write } t = g(u) \rightarrow c. \quad \lim_{t \rightarrow c} f(t) = f(c).$$

Some books define  $e^r = \lim_{x \rightarrow \infty} \left(1 + \frac{r}{x}\right)^x$ .

This limit comes from compound interest.

If you deposit a principal amount  $A$  in the bank at nominal interest rate  $r$  (eg. 5% interest per annum would give  $r = 0.05$ ). If interest is compounded annually then after one year you have earned  $rA$  interest. The total balance in the bank after a year would be  $A + rA = (1+r)A$ .

If interest is compounded semiannually (every 6 months) then after 6 months you have  $(1 + \frac{r}{2})A$  as balance after 6 months; then at the end of the year you have  $(1 + \frac{r}{2}) \cdot (1 + \frac{r}{2})A = (1 + \frac{r}{2})^2 A = (1 + r + \frac{r^2}{4})A$ .

If interest is compounded  $n$  times per year then every  $\frac{1}{n}$  year your balance is multiplied by  $1 + \frac{r}{n}$ . After one year your balance is  $(1 + \frac{r}{n})^n A$ .

For continually compounded interest we let  $n \rightarrow \infty$ .

$\lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^n A = e^r A$ . Eg. 5% interest compounded continuously results in

balance  $e^{0.05} A \approx 1.05127 A$  (effectively 5.127% interest rate per annum).

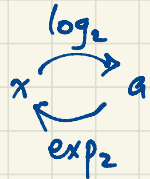
p. 31 # 69.  $\lim_{x \rightarrow \infty} \frac{\log_2 x}{\log_3 x} = \lim_{x \rightarrow \infty} \frac{(\ln x) / \ln 2}{(\ln x) / \ln 3} = \frac{\ln 3}{\ln 2} = \log_2 3$

$\log_2 x = \frac{\ln x}{\ln 2}$  why?

$a = \log_2 x \iff 2^a = x \iff \ln(2^a) = \ln x$

$\iff a \ln 2 = \ln x$

$\iff a = \frac{\ln x}{\ln 2}$



April 21

### Sec 4.6 Linearization and Differentials

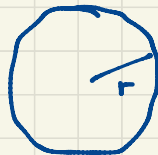
$y = f(x)$

$\frac{dy}{dx} = f'(x)$

$\Delta y \approx dy = f'(x) dx = f'(x) \Delta x$

small change in  $y$   
due to change  
in  $x$ .

Formulas for area and circumference of a circle

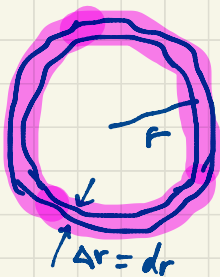


$A = \pi r^2$

$C = 2\pi r$

From the formula  $A = \pi r^2$  you can deduce the formula  $C = 2\pi r$  using linearization.

Starting with a circle of radius  $r$ , area  $A = \pi r^2$ , enlarge the circle by expanding the radius by a small amount  $\Delta r$ .



$$\Delta A \approx C \Delta r \approx C \Delta r$$

$$\Delta A \approx dA = 2\pi r dr = 2\pi r \Delta r$$

$$A = \pi r^2$$

$$\frac{dA}{dr} = 2\pi r$$

$$dA = 2\pi r dr$$

$$\text{So } C = 2\pi r.$$

For a sphere of radius  $r$ ,

$$V = \frac{4}{3}\pi r^3. \quad \text{What is the surface area?}$$

Imagine painting the surface of a ball of radius  $r$ . How much paint is required to paint the surface? If I add a layer of paint of thickness  $\Delta r$  then the new volume changes by

$$\Delta V \approx A \Delta r$$

$$\Delta V \approx dV = 4\pi r^2 dr = 4\pi r^2 \Delta r$$

$$\text{So } A = 4\pi r^2.$$

Sec 4.6 p. 299 #28.  $f(x) = \frac{x}{x+1}$ . Approximate  $f(1.1)$  using the linearization at  $x=1$ .

$f(1) = \frac{1}{2} = 0.5$ . This value is the base for our linearization.

$$L(x) = f(a) + f'(a)(x-a), \quad a=1$$
$$= \frac{1}{2} + \frac{1}{4}(x-1)$$

$$f(1.1) \approx L(1.1) = \frac{1}{2} + \frac{1}{4}(0.1) = 0.5 + 0.025 = 0.525$$

$$\frac{dy}{dx} = f'(x) = \frac{(x+1) \cdot 1 - x \cdot 1}{(x+1)^2}$$

$$= \frac{1}{(x+1)^2}$$

$$dy = \frac{dx}{(x+1)^2}$$

Alternatively using differentials:

$x$  goes from 1 to 1.1, a small change of  $\Delta x = dx = 0.1$ . What is the corresponding change in  $y$ ?

$$\Delta y \approx dy = \frac{dx}{(x+1)^2}$$

$$\Delta y \approx \frac{\Delta x}{(1+1)^2} = \frac{\Delta x}{4} = \frac{0.1}{4} = 0.025$$

$$y \approx 0.5 + \Delta y = 0.525.$$

$$f(1.1) = \frac{1.1}{2.1} = 0.52380952\dots$$

The error is  $0.525 - f(1.1) = 0.00119\dots$

The relative error is  $\frac{0.00119}{0.5238} = 0.0023\dots$

Actually  $0.0023 = 0.23\%$

"per cent" means "divided by 100"

(an over estimate since this difference is positive).

The percentage error is  $0.0023 \times 100\% = 0.23\%$   
(about a quarter of a percent error).

## Sec 4.9 Antiderivatives

$$\frac{d}{dx} (7x^2 + 2x + 1)^3 = 3(7x^2 + 2x + 1)^2 (14x + 2).$$

Reverse: ~~The~~ <sup>An</sup> antiderivative of  $3(7x^2 + 2x + 1)^2 (14x + 2)$  is  $(7x^2 + 2x + 1)^3$ .

The general answer for antiderivative:  $(7x^2 + 2x + 1)^3 + \text{constant}$ .

$$\frac{d}{dx} [(7x^2 + 2x + 1)^3 + 11] = 3(7x^2 + 2x + 1)^2 (14x + 2).$$

We write  $\int 3(7x^2 + 2x + 1)^2 (14x + 2) dx = (7x^2 + 2x + 1)^3 + C$  where  $C$  is an arbitrary

The general antiderivative of  $3(7x^2 + 2x + 1)^2 (14x + 2)$  with constant respect to  $x$  is  $(7x^2 + 2x + 1)^3 + \text{constant}$ .

April 22

$$\text{eg. } \int x^k dx = \begin{cases} \frac{1}{k+1} x^{k+1}, & \text{if } k \neq -1 \\ \ln|x|, & \text{if } k = -1 \end{cases}$$

$$\text{Check: } \frac{d}{dx} \frac{x^{k+1}}{k+1} = \frac{(k+1)}{k+1} x^k = x^k \quad (k \text{ constant}).$$

$$\frac{d}{dx} (\ln|x|) = x^{-1} = \frac{1}{x}$$

$$\int (x^3 - 7x^2 - 5x + 3) dx = \frac{1}{4}x^4 - \frac{7}{3}x^3 - \frac{5}{2}x^2 + 3x + C$$

$$\text{Check: } \frac{d}{dx} \left( \frac{1}{4}x^4 - \frac{7}{3}x^3 - \frac{5}{2}x^2 + 3x + C \right) = x^3 - 7x^2 - 5x + 3$$

Compare notation:  $\int f(x) dx$  antiderivative (function) (indefinite integral)

vs.  $\int_a^b f(x) dx$  integral (a number) (definite integral)

$\uparrow$  integrand

How do we find antiderivatives? Best method: guess and check.

$$\text{Eq. } \int x \sin x dx = \sin x - x \cos x + C$$

$$\frac{d}{dx} (?) = x \sin x$$

$$\frac{d}{dx} (x \cos x) = 1 \cdot \cos x + x \cdot (-\sin x) = \cos x - x \sin x$$

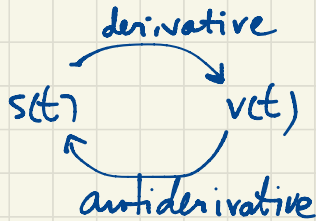
$$\frac{d}{dx} (-\cos x) = \sin x$$

$$\frac{d}{dx} (x \cos x - \sin x) = (\cancel{\cos x} - x \sin x) - \cancel{\cos x} = -x \sin x$$

$$\frac{d}{dx} (\sin x - x \cos x) = x \sin x$$

Sec 4.9 #92. Given velocity  $v(t) = e^t + 4$  } initial value problem  
 and initial position  $s(0) = 2$  }  
 find  $s(t)$ , the position at time  $t$ .

Recall  $s'(t) = v(t)$  so  $s(t) = \int v(t) dt = \int (e^t + 4) dt = e^t + 4t + C$



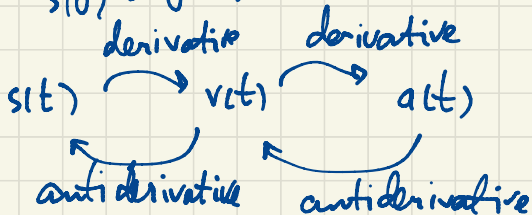
Using  $s(0) = 2$ , find  $C$ .

$$s(t) = e^t + 4t + C$$

$$2 = s(0) = 1 + 0 + C \Rightarrow C = 1$$

So  $s(t) = e^t + 4t + 1$ .

#97.  $a(t) = -32$  } differential equation } initial value problem  
 $v(0) = 20$  } initial conditions }  
 $s(0) = 0$  }



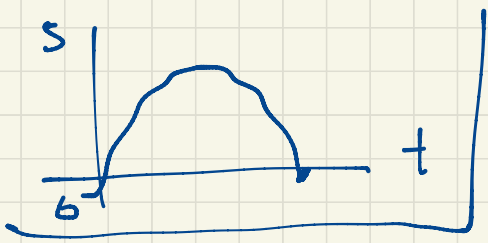
$$v(t) = \int a(t) dt = \int (-32) dt = -32t + C$$

$$20 = v(0) = C \Rightarrow v(t) = -32t + 20$$

$$s(t) = \int v(t) dt = \int (-32t + 20) dt = -16t^2 + 20t + K$$

Since  $s(0) = 0 = K$ ,

$s(t) = -16t^2 + 20t$ .



#100.  $a(t) = 2 \cos t$

$$v(0) = 1$$

$$s(0) = 0$$

$$v(t) = \int a(t) dt = \int 2 \cos t dt = 2 \sin t + C$$

$$1 = v(0) = C \Rightarrow v(t) = 2 \sin t + 1$$

$$s(t) = \int v(t) dt = \int (2 \sin t + 1) dt$$

$$= -2 \cos t + t + K$$

$$0 = s(0) = -2 + K \Rightarrow K = 2$$

$$s(t) = -2 \cos t + t + 2$$

$$\int \frac{dt}{t^2+1} = \int \frac{1}{t^2+1} dt = \tan^{-1} t + C$$

~~$$\int \tan^{-1} t dt = \frac{1}{t^2+1}$$~~

$$\frac{d}{dt} \frac{1}{t^2+1} = -\frac{1}{(t^2+1)^2} \cdot 2t = -\frac{2t}{(t^2+1)^2}$$

$$\int \frac{dt}{t} = \int \frac{1}{t} dt = \ln |t| + C$$

$$\int dt = \int 1 dt = t + C$$