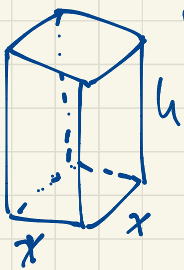
Math 2200-01 (Calculus I) Spring 2020

Book 3

Sec 4.5: Optimization

April 13

p.285 #19. Of all boxes with a square base and a volume $8\,m^3$, which one has the minimum surface area?



volume $V = x^2 h = 8 \Rightarrow h = \dfrac{8}{x^2}$

area $A = \underbrace{2x^2}_{\text{top and bottom}} + \underbrace{4xh}_{\text{sides}} = 2x^2 + 4x \cdot \dfrac{8}{x^2} = 2x^2 + \dfrac{32}{x}, \quad x > 0$

The domain is $(0, \infty)$, an unbounded open interval.

$$\frac{dA}{dx} = 4x - \frac{32}{x^2} = \frac{4}{x^2}\left(x^3 - 8\right)$$

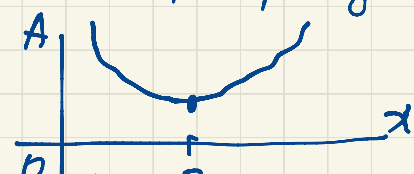
The critical point is at $x = 2$.

(Recall: critical points are where the derivative is zero or undefined. There are no points of the domain where $\frac{dA}{dx}$ is undefined, only one point where $\frac{dA}{dx} = 0$.)

For $0 < x < 2$, $\frac{dA}{dx} < 0$ so $A(x)$ is decreasing.

For $x = 2$, $\frac{dA}{dx} = 0$.

For $x > 2$, $\frac{dA}{dx} > 0$ so $A(x)$ is increasing.

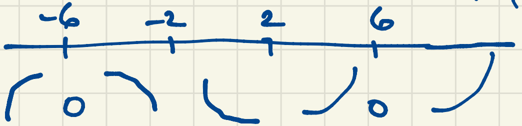


So the minimum surface area $A(2) = 12\,m^2$ occurs for a box of size $2m \times 2m \times 2m$.

Sec 4.4. p.278 #21. $f(x) = (x-6)(x+6)^2 = (x^2-36)(x+6) = x^3+6x^2-36x-216$

$f'(x) = 3x^2+12x-36 = 3(x^2+4x-12) = 3(x+6)(x-2)$

$f''(x) = 6x+12 = 6(x+2)$

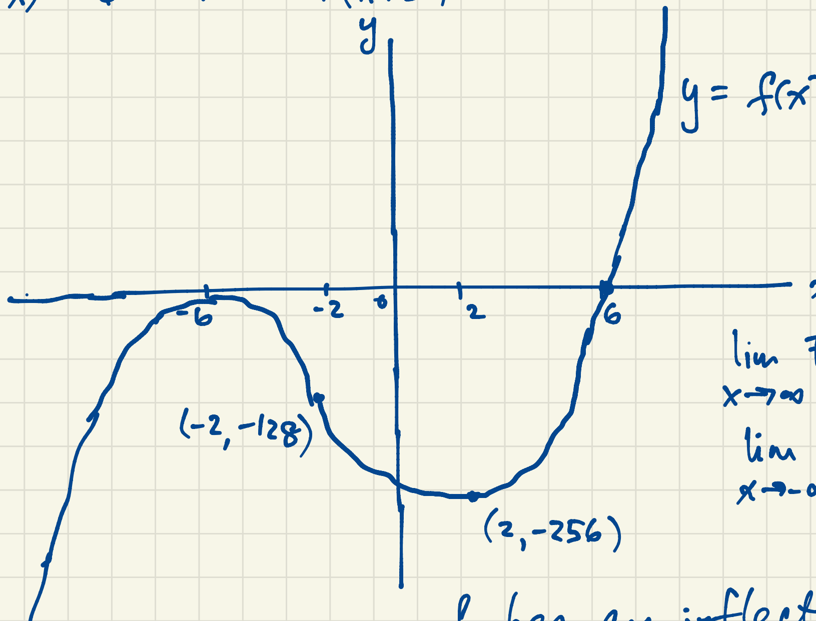


f' + + 0 − − − − 0 + + + +

f'' − − − − 0 + + + + + + +

$f(-2) = (-8)(4)^2 = -128$

$f(2) = (-4)(8^2) = -256$



$\lim_{x\to\infty} f(x) = \infty$

$\lim_{x\to-\infty} f(x) = -\infty$

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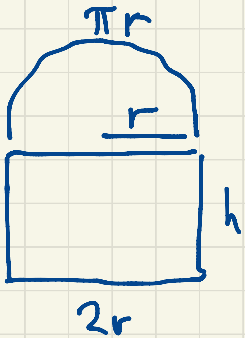
f is increasing on $(-\infty,-6)$ and on $(2,\infty)$, decreasing on $(-6,2)$, concave down on $(-\infty,-2)$, concave up on $(-2,\infty)$.

f has an inflection point $(-2,-128)$, a local minimum point $(2,-256)$, a local maximum point $(-6,0)$, no absolute extrema or asymptotes.

Sec 4.5 p. 287 #41. Let r be the radius of the semicircular window pane.

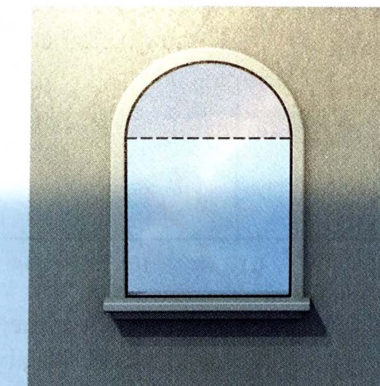


The perimeter is $P = \pi r + 2r + 2h = 20$

$$(\pi+2)r + 2h = 20 \qquad h = 10 - \frac{\pi+2}{2}r$$

$$2h = 20 - \pi r - 2r$$

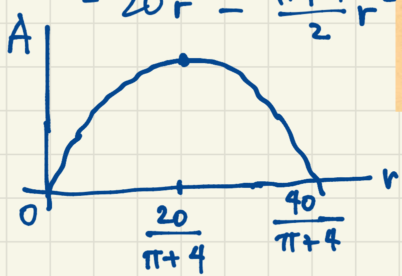
41. A window consists of rectangular pane of glass surmounted by a semicircular pane of glass (see figure). If the perimeter of the window is 20 feet, determine the dimensions of the window that maximize the area of the window.



$$\frac{dA}{dr} = 20 - (\pi+4)r$$

$$A = \frac{\pi}{2}r^2 + 2rh$$
$$= \frac{\pi}{2}r^2 + 2r\left(10 - \frac{\pi+2}{2}r\right)$$
$$= 20r + \left(\frac{\pi}{2} - (\pi+2)\right)r^2$$
$$= 20r - \frac{\pi+4}{2}r^2$$

$$A = \left(20 - \frac{\pi+4}{2}r\right)r$$



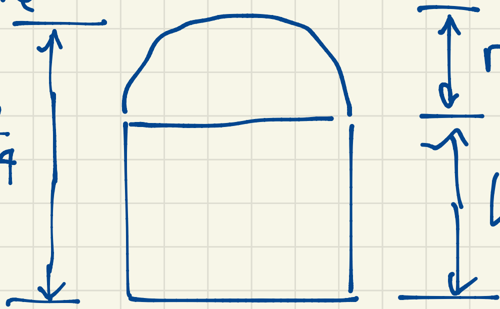
The critical point is at $r = \frac{20}{\pi+4}$.

When $0 < r < \frac{20}{\pi+4}$, $\frac{dA}{dr} > 0$ so A is increasing.

When $\frac{20}{\pi+4} < r < \frac{40}{\pi+4}$, $\frac{dA}{dr} < 0$ so A is decreasing.

So the maximum area occurs when $r = \frac{20}{\pi+4}$. Alternatively since $A \geq 0$ requires r to be in $\left[0, \frac{40}{\pi+4}\right]$, we need only check A at endpoints and the critical point.

The dimensions of the window that maximizes the area are



$\frac{40}{\pi+4}$

$r = \frac{20}{\pi+4}$

$$h = 10 - \frac{\pi+2}{2} r = 10 - \frac{\pi+2}{2} \cdot \frac{20}{\pi+4}$$

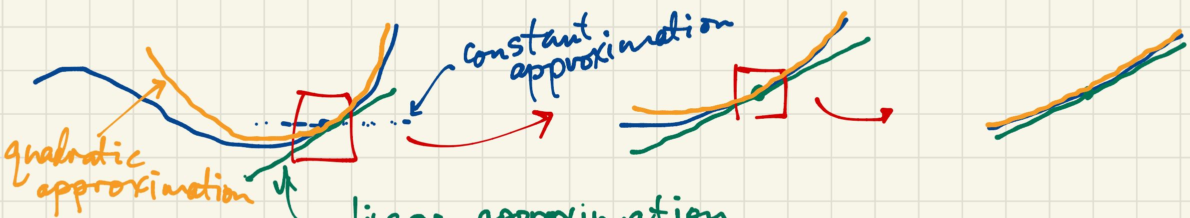
$$= 10 - \frac{10(\pi+2)}{\pi+4}$$

$$= \frac{10(\pi+4) - 10(\pi+2)}{\pi+4}$$

$$= \frac{20}{\pi+4}$$

$2r = \frac{40}{\pi+4}$

Sec 4.6. Linearization and Differentials



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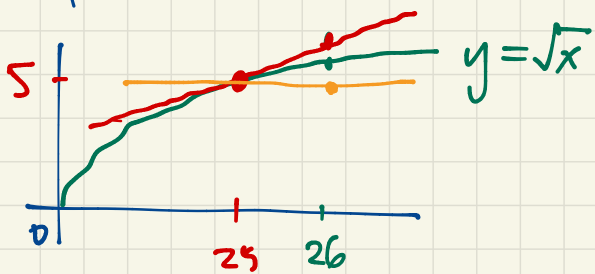
For x close to a, $f(x) \approx L(x) = f'(a)(x-a) + f(a) = \boxed{}\,x + \boxed{}$

approximately equal to

$L(x)$: linearization of f at $(a, f(a))$

$f'(a)$ $\qquad$ $f(a) - af'(a)$

Example: Use the linearization of $\sqrt{x}$ at $(25, 5)$ to approximate $\sqrt{26}$.



$f(x) = \sqrt{x}$, $\quad f(25) = 5$

$f'(x) = \frac{1}{2\sqrt{x}}$, $\quad f'(25) = \frac{1}{2\sqrt{25}} = \frac{1}{10}$

The linearization of $\sqrt{x}$ at $(25, 5)$ is

$$L(x) = f'(25)(x-25) + f(25)$$
$$= \frac{1}{10}(x-25) + 5.$$

If $x \approx 25$ then $\sqrt{x} \approx \frac{1}{10}(x-25) + 5$.

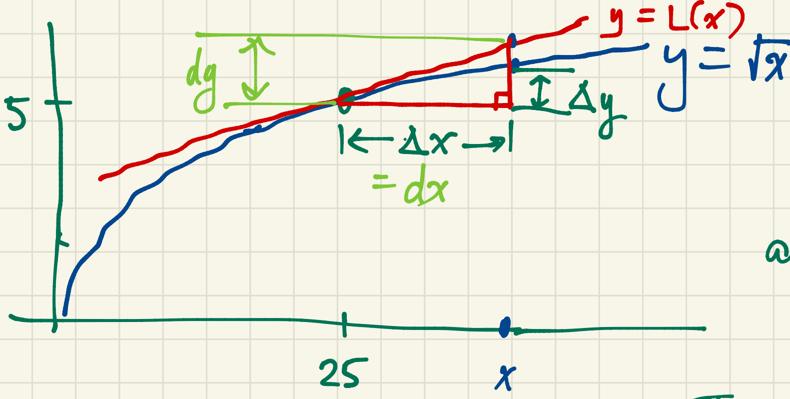
Eg. $\sqrt{26} \approx \frac{1}{10}(26-25)+5 = 5.1$

Check: $\sqrt{26} \approx 5.099019514$

correct to 3 significant digits.

Cruder approximation: $\sqrt{26} \approx 5$. (correct to one significant digit)

Constant approximation $\sqrt{x} \approx 5$



If we move from $(25, 5)$ to $(x, f(x))$ on the graph, our actual function $f(x) = \sqrt{x}$ changes by an exact amount

$$\Delta y = f(x) - f(25).$$

The secant line from $(25, 5)$ to $(x, f(x))$ has slope

$$\frac{\Delta y}{\Delta x} = \frac{f(x) - f(25)}{x - 25}.$$

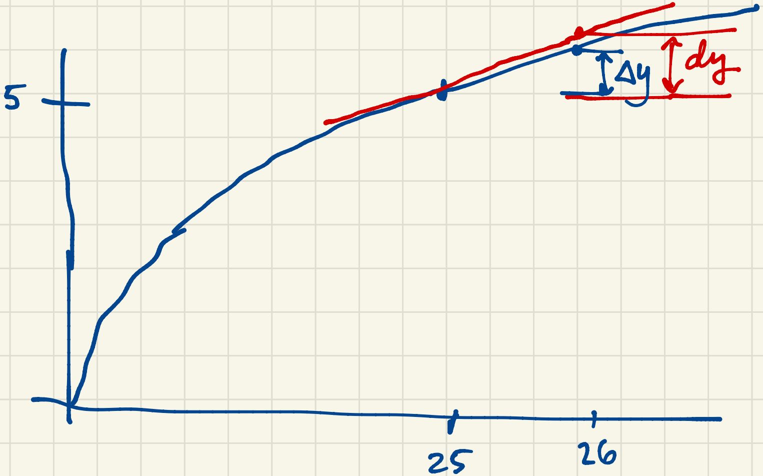
On the tangent line,

$$\frac{\Delta L(x)}{\Delta x} = \frac{L(x) - L(25)}{x - 25} = f'(25) = \frac{dy}{dx}$$

Until now we have written $\frac{dy}{dx}$ as an indivisible symbol. Now we are interpreting dx and dy as changes in x and y. They are changes on the tangent line (just like Δx and Δy are corresponding changes on the actual curve of f).

dx and dy are <u>differentials</u>.

We'll interpret $\sqrt{26} \approx 5.1$ in this new language:



$f'(25) = \frac{1}{10} = \frac{dy}{dx} \Rightarrow dy = \frac{1}{10} dx = \frac{1}{10} \times 1 = 0.1$

$dx = \Delta x = 26 - 25 = 1$

As we move from $x = 25$ to $x = 26$,
the corresponding change in y is

$\Delta y \approx dy = 0.1$.

So $\sqrt{26} \approx 5.1$

This is a quick and easy interpretation for differentials. We will be using differentials throughout Calculus.

Integrals $\int_a^b f(x)\, \underbrace{dx}$

$x \longrightarrow u \longrightarrow y$

$\frac{dy}{dx} = \frac{dy}{\cancel{du}} \cdot \frac{\cancel{du}}{dx}$

If $y = \sin x$, find $\frac{dy}{dx}$ and dy.

$\frac{dy}{dx} = \cos x$ so $dy = (\cos x)\, dx$

so $\Delta y \approx dy = \cos x\, dx$

Sec 4.7 l'Hôpital's Rule

The limit $\lim\limits_{x\to\infty} \underbrace{\dfrac{x^3+2x^2-x-1}{2x^3-5x}}_{\text{"indeterminate form } \frac{\infty}{\infty}\text{"}} = \lim\limits_{x\to\infty} \underbrace{\dfrac{1+\frac{2}{x}-\frac{1}{x^2}-\frac{1}{x^3}}{2-\frac{5}{x^2}}}_{\text{determinate form}} = \dfrac{1}{2}$.

The limit $\lim\limits_{x\to\infty} \dfrac{1}{3x^2+5} = 0$.

"determinate form $\frac{1}{\infty}$"

Do not confuse l'Hôpital's Rule with the Quotient Rule!

$$\lim_{x\to 0} \underbrace{\frac{\sin x}{x}}_{\text{indeterminate form } \frac{0}{0}} = 1$$

l'Hôpital's Rule For limits of indeterminate form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, $\lim\limits_{x} \dfrac{f(x)}{g(x)} = \lim\limits_{x} \dfrac{f'(x)}{g'(x)}$ assuming the second limit exists.

eg. $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = \lim\limits_{x\to 0} \dfrac{\cos x}{1} = 1$.

eg. $\lim\limits_{x\to\infty} \dfrac{x^3+2x^2-x-1}{2x^3-5x} = \lim\limits_{x\to\infty} \dfrac{3x^2+4x-1}{6x^2-5} = \lim\limits_{x\to\infty} \dfrac{6x+4}{12x} = \lim\limits_{x\to\infty} \dfrac{6}{12} = \dfrac{1}{2}$

eg. $\lim_{x\to\infty} \underline{xe^{-x}} = \lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0$

indeterminate form $\infty\cdot 0$ indeterminate form $\frac{\infty}{\infty}$

eg. $\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} (-x) = 0.$

indeterminate form $0\cdot(-\infty)$ indeterminate $\frac{-\infty}{\infty}$

Don't keep using l'Hôpital's Rule beyond this point; that approach never reaches a conclusion.

$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{x}{\frac{1}{\ln x}} = \lim_{x\to 0^+} \frac{1}{-\frac{1}{(\ln x)^2}\cdot\frac{1}{x}} = \lim_{x\to 0^+} \left(-x(\ln x)^2\right)$ this is not an improvement!

indeterminate form $\frac{0}{0}$

$(a-b)(a+b) = a^2-b^2$

$\lim_{x\to\infty} \left(\sqrt{x^2+10x} - x\right)\cdot\frac{\sqrt{x^2+10x}+x}{\sqrt{x^2+10x}+x} = \lim_{x\to\infty} \frac{x^2+10x-x^2}{\sqrt{x^2+10x}+x} = \lim_{x\to\infty} \frac{10x}{\sqrt{x^2+10x}+x}$

indeterminate form $\infty-\infty$

$= \lim_{x\to\infty} \frac{10}{\sqrt{1+\frac{10}{x}}+1} = \frac{10}{\sqrt{1+0}+1} = 5.$

Note:

$\sqrt{x^2+10x} = \sqrt{x^2\left(1+\frac{10}{x}\right)}$

$= x\sqrt{1+\frac{10}{x}}$

$\sqrt{x^2} = |x| = \begin{cases} x & \text{if } x\geq 0 \\ -x & \text{if } x<0 \end{cases}$

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Another l'Hôpital's Rule example for the indeterminate form 1^∞.

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x \quad \text{where } a \text{ is constant.}$$

In order to use l'Hôpital's Rule, we need to rewrite this as a quotient.

$$b^c = (e^{\ln b})^c = e^{c\ln b}$$

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x = \lim_{x\to\infty} e^{x\ln\left(1+\frac{a}{x}\right)} = \lim_{u\to 1} e^{\frac{a}{u-1}\ln u} = e^{a\cdot 1} = e^a.$$

$$u = 1+\frac{a}{x} \longrightarrow 1.$$

$$u-1 = \frac{a}{x}$$

$$x = \frac{a}{u-1}$$

$$\lim_{u\to 1}\frac{\ln u}{u-1} = \lim_{u\to 1}\frac{1/u}{1} = 1.$$

(indeterminate form $\frac{0}{0}$)

If $f(x) = e^{ax}$ then $\lim_{u\to 1} f\left(\frac{\ln u}{u-1}\right) = f\left(\lim_{u\to 1}\frac{\ln u}{u-1}\right) = f(1) = e^{a\cdot 1} = e^a.$

Does $\lim_{u} f(g(u)) = f\left(\lim_{u} g(u)\right)$? Can you move the limit inside?

We can do this when f is continuous and g is continuous.

$c = \lim_{u} g(u)$, write $t = g(u) \to c$. $\lim_{t\to c} f(t) = f(c)$.

Some books define $e^a = \lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x$.

This limit comes from compound interest.

If you deposit a principal amount A in the bank at nominal interest rate r (eg. 5% interest per annum would give $r=0.05$). If interest is compounded annually then after one year you have earned rA interest. The ~~total~~ balance in the bank after a year would be
$A + rA = (1+r)A$.

If interest is compounded semiannually (every 6 months) then after 6 months you have $(1+\frac{r}{2})A$ as balance after 6 months; then at the end of the year you have

$$\left(1+\frac{r}{2}\right)\cdot\left(1+\frac{r}{2}\right)A = \left(1+\frac{r}{2}\right)^2 A = \Big(1+\underbrace{r+\frac{r^2}{4}}_{\text{effective rate of interest.}}\Big)A.$$

If interest is compounded n times per year then every $\frac{1}{n}$ year your balance is multiplied by $1+\frac{r}{n}$. After one year your balance is $\left(1+\frac{r}{n}\right)^n A$.

For continually compounded interest we let $n\to\infty$.

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n A = e^r A.$$

Eg. 5% interest compounded continuously results in balance $e^{0.05}A \approx 1.05127A$ (effectively 5.127 % interest rate per annum).

p. 311 # 69. $\lim_{x\to\infty} \frac{\log_2 x}{\log_3 x} = \lim_{x\to\infty} \frac{(\cancel{\ln x})/\ln 2}{(\cancel{\ln x})/\ln 3} = \frac{\ln 3}{\ln 2} = \log_2 3$

$\log_2 x = \frac{\ln x}{\ln 2}$ why?

$a = \log_2 x \iff 2^a = x \iff \ln(2^a) = \ln x$

$\iff a \ln 2 = \ln x$

$\iff a = \frac{\ln x}{\ln 2}.$

$$\begin{tikzcd} x \arrow[r, bend left, "\log_2"] & a \arrow[l, bend left, "\exp_2"] \end{tikzcd}$$

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Sec 4.6 Linearization and Differentials

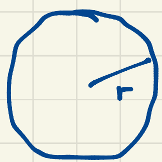
$y = f(x)$

$\frac{dy}{dx} = f'(x)$

$\Delta y \approx dy = f'(x)\,dx = f'(x)\Delta x$

small change in y due to change in x.

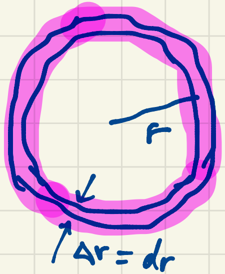
Formulas for area and circumference of a circle



$A = \pi r^2$

$C = 2\pi r$

From the formula $A = \pi r^2$ you can deduce the formula $C = 2\pi r$ using linearization.



Starting with a circle of radius r, area $A = \pi r^2$, enlarge the circle by expanding the radius by a small amount Δr.

$$\Delta A \approx C\,\Delta r \approx C\,\Delta r$$

$$\Delta A \approx dA = 2\pi r\,dr = 2\pi r\,\Delta r$$

$$A = \pi r^2$$

$$\frac{dA}{dr} = 2\pi r$$

$$dA = 2\pi r\,dr$$

So $C = 2\pi r$.

For a sphere of radius r,

$V = \frac{4}{3}\pi r^3$. What is the surface area?

Imagine painting the surface of a ball of radius r. How much paint is required to paint the surface? If I add a layer of paint of thickness Δr then the new volume changes by

$$\Delta V \approx A\,\Delta r$$

$$\Delta V \approx dV = 4\pi r^2\,dr = 4\pi r^2\,\Delta r$$

So $A = 4\pi r^2$.

Sec 4.6 p. 299 #28. $f(x) = \frac{x}{x+1}$. Approximate $f(1.1)$ using the linearization at $x=1$.

$f(1) = \frac{1}{2} = 0.5$. This value is the base for our linearization.

$$\frac{dy}{dx} = f'(x) = \frac{(x+1)\cdot 1 - x\cdot 1}{(x+1)^2} = \frac{1}{(x+1)^2}$$

$$dy = \frac{dx}{(x+1)^2}$$

$L(x) = f(a) + f'(a)(x-a)$, $a=1$

$= \frac{1}{2} + \frac{1}{4}(x-1)$

$f(1.1) \approx L(1.1) = \frac{1}{2} + \frac{1}{4}(0.1) = 0.5 + 0.025 = 0.525$

Alternatively using differentials:

x goes from 1 to 1.1, a small change of $\Delta x = dx = 0.1$. What is the corresponding change in y?

$$\Delta y \approx dy = \frac{dx}{(x+1)^2}$$

$$\Delta y \approx \frac{\Delta x}{(1+1)^2} = \frac{\Delta x}{4} = \frac{0.1}{4} = 0.025$$

$$y \approx 0.5 + \Delta y = 0.525.$$

Actually $0.0023 = 0.23\%$

"per cent" means "divided by 100"

$f(1.1) = \frac{1.1}{2.1} = 0.52380952\ldots$

The error is $0.525 - f(1.1) = 0.00119\ldots$ (an over estimate since this difference is positive).

The relative error is $\frac{0.00119}{0.5238} = 0.0023\ldots$ The percentage error is $0.0023 \times 100\% = 0.23\%$ (about a quarter of a percent error).

Sec 4.9 Antiderivatives

$$\frac{d}{dx}(7x^2+2x+1)^3 = 3(7x^2+2x+1)^2(14x+2).$$

Reverse: ~~The~~ An antiderivative of $3(7x^2+2x+1)^2(14x+2)$ is $(7x^2+2x+1)^3$.

The general answer for antiderivative: $(7x^2+2x+1)^3$ + constant.

$$\frac{d}{dx}\left[(7x^2+2x+1)^3+11\right] = 3(7x^2+2x+1)^2(14x+2).$$

We write $\int 3(7x^2+2x+1)^2(14x+2)\,dx = (7x^2+2x+1)^3+C$ where C is an arbitrary constant.

The general antiderivative of $3(7x^2+2x+1)^2(14x+2)$ with respect to x is $(7x^2+2x+1)^3$ + constant.

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eg. $\int x^k\,dx = \begin{cases} \frac{1}{k+1}x^{k+1}, & \text{if } k \neq -1 \\ \ln|x|, & \text{if } k=-1 \end{cases}$

Check: $\frac{d}{dx}\frac{x^{k+1}}{k+1} = \frac{\cancel{(k+1)}x^k}{\cancel{k+1}} = x^k$ (k constant).

$$\frac{d}{dx}(\ln|x|) = x^{-1} = \frac{1}{x}$$

$$\int (x^3 - 7x^2 - 5x + 3)\, dx = \frac{1}{4}x^4 - \frac{7}{3}x^3 - \frac{5}{2}x^2 + 3x + C$$

Check: $\frac{d}{dx}\left(\frac{1}{4}x^4 - \frac{7}{3}x^3 - \frac{5}{2}x^2 + 3x + C\right) = x^3 - 7x^2 - 5x + 3$

Compare notation: $\int f(x)\, dx$ antiderivative (function) (indefinite integral)

vs. $\int_a^b \underbrace{f(x)}_{\text{integrand}}\, dx$ integral (a number) (definite integral)

How do we find antiderivatives? Best method: guess and check.

Eg. $\int x \sin x \, dx = \sin x - x\cos x + C$

$\frac{d}{dx}(?) = x \sin x$

$\frac{d}{dx}(-\cos x) = \sin x$

$\frac{d}{dx}(x\cos x) = 1\cdot \cos x + x\cdot(-\sin x) = \cos x - x\sin x$

$\frac{d}{dx}(x\cos x - \sin x) = (\cancel{\cos x} - x\sin x) - \cancel{\cos x}$

$= -x\sin x$

$\frac{d}{dx}(\sin x - x\cos x) = x \sin x$

Sec 4.9 #92. Given velocity $v(t) = e^t + 4$ and initial position $s(0) = 2$ } initial value problem

find $s(t)$, the position at time t.

Recall $s'(t) = v(t)$ so $s(t) = \int v(t)\,dt = \int (e^t + 4)\,dt = e^t + 4t + C$

$$\begin{tikzcd} s(t) \arrow[rr, bend left, "\text{derivative}"] & & v(t) \arrow[ll, bend left, "\text{antiderivative}"] \end{tikzcd}$$

Using $s(0) = 2$, find C.

$$s(t) = e^t + 4t + C$$

$$2 = s(0) = 1 + 0 + C \Rightarrow C = 1$$

So $s(t) = e^t + 4t + 1$.

#97. $a(t) = -32$ } differential equation

$v(0) = 20$

$s(0) = 0$ } initial conditions

} initial value problem

$$\begin{tikzcd} s(t) \arrow[r, bend left, "\text{derivative}"] & v(t) \arrow[l, bend left, "\text{antiderivative}"] \arrow[r, bend left, "\text{derivative}"] & a(t) \arrow[l, bend left, "\text{antiderivative}"] \end{tikzcd}$$

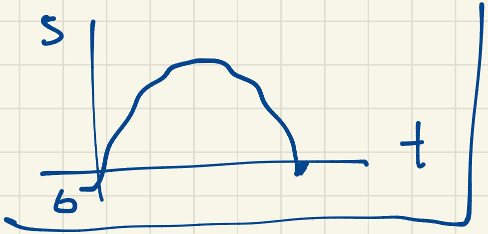
$$v(t) = \int a(t)\,dt = \int (-32)\,dt = -32t + C$$

$$20 = v(0) = C \Rightarrow v(t) = -32t + 20$$

$$s(t) = \int v(t)\,dt = \int (-32t + 20)\,dt$$

$$= -16t^2 + 20t + K.$$ Since $s(0) = 0 = K$,

$s(t) = -16t^2 + 20t$.



#100. $a(t) = 2\cos t$

$v(0) = 1$

$s(0) = 0$

$$v(t) = \int a(t)\,dt = \int 2\cos t\,dt = 2\sin t + C$$

$$1 = v(0) = C \implies v(t) = 2\sin t + 1$$

$$s(t) = \int v(t)\,dt = \int (2\sin t + 1)\,dt$$

$$= -2\cos t + t + K$$

$$0 = s(0) = -2 + K \implies K = 2$$

$$s(t) = -2\cos t + t + 2$$

$$\int \frac{dt}{t^2+1} = \int \frac{1}{t^2+1}\,dt = \tan^{-1} t + C$$

$$\cancel{\int \tan^{-1} t\,dt = \frac{1}{t^2+1}}$$

$$\int \frac{dt}{t} = \int \frac{1}{t}\,dt = \ln|t| + C$$

$$\int dt = \int 1\,dt = t + C$$

$$\frac{d}{dt}\frac{1}{t^2+1} = -\frac{1}{(t^2+1)^2}\cdot 2t = -\frac{2t}{(t^2+1)^2}$$

April 24

Brushing up on l'Hôpital's Rule

$\lim_{x\to\infty} \frac{x^n}{e^x} = 0$ (n applications of l'Hôpital's Rule)

Eg for $n=4$, $\lim_{x\to\infty} \frac{x^4}{e^x} = \lim_{x\to\infty} \frac{4x^3}{e^x} = \lim_{x\to\infty} \frac{12x^2}{e^x} = \lim_{x\to\infty} \frac{24x}{e^x} = \lim_{x\to\infty} \frac{24}{e^x} = 0.$

$\lim_{x\to\infty} \frac{x^4}{e^{x^2}} = \lim_{x\to\infty} \frac{4x^3}{2xe^{x^2}} = \lim_{x\to\infty} \frac{2x^2}{e^{x^2}} = \cdots = 0.$

$e^{x^2} = e^{(x^2)}$ or $\cancel{(e^x)^2}$

$(e^x)^2 = e^{2x}$

$\lim_{x\to\infty} \frac{x^4}{e^{x^2}} = \lim_{u\to\infty} \frac{u^2}{e^u} = 0$

$u = x^2$

OR: $0 < \frac{x^4}{e^{x^2}} < \frac{x^4}{e^x}$, $\;\downarrow 0$

$\lim_{x\to\infty} \frac{x^4}{e^{x^2}} = 0$ by the Squeeze Theorem.

p. 311 # 28. $\lim_{x\to 0^+} \frac{x - 3\sqrt{x}}{x - \sqrt{x}} = \lim_{x\to 0^+} \frac{1 - \frac{3}{2\sqrt{x}}}{1 - \frac{1}{2\sqrt{x}}} = \lim_{x\to 0^+} \frac{2\sqrt{x} - 3}{2\sqrt{x} - 1} = 3.$

OR $\lim_{x\to 0^+} \frac{x - 3\sqrt{x}}{x - \sqrt{x}} = \lim_{x\to 0^+} \frac{\sqrt{x} - 3}{\sqrt{x} - 1} = \frac{0-3}{0-1} = 3.$

#40. $\lim_{x\to 0} \dfrac{\sin x - x}{7x^3} = \lim_{x\to 0} \dfrac{\cos x - 1}{21x^2} = \lim_{x\to 0} \dfrac{-\sin x}{42x} = \lim_{x\to 0} \dfrac{-\cos x}{42} = -\dfrac{1}{42}$

#83. $\lim_{x\to 0} (x+\cos x)^{1/x} = \lim_{x\to 0} e^{\frac{1}{x}\ln(x+\cos x)} = e^1 = e$.

indeterminate form 1^∞

using $e^{\ln a} = a$, $e^{\ln(x+\cos x)} = (x+\cos x)$

Compare: $\lim_{x\to 0^+} (x+1)^{1/x} = \lim_{t\to\infty} \left(1+\frac{1}{t}\right)^t = e$

$t = \frac{1}{x} \to \infty$

$\lim_{x\to 0} \dfrac{\ln(x+\cos x)}{x} = \lim_{x\to 0} \dfrac{\frac{1}{x+\cos x}(1-\sin x)}{1} = 1$

indeterminate form $\frac{0}{0}$

#68. $\lim_{x\to 0^+} x^{\frac{1}{\ln x}} = \lim_{x\to 0^+} e^{\frac{1}{\ln x}\ln x} = \lim_{x\to 0^+} e = e$.

$x = e^{\ln x} \Rightarrow x^{\frac{1}{\ln x}} = \left(e^{\ln x}\right)^{\frac{1}{\ln x}} = e$

#20. $\lim_{x\to 0} \dfrac{e^x - 1}{2x+5} = \dfrac{0}{5} = 0$.

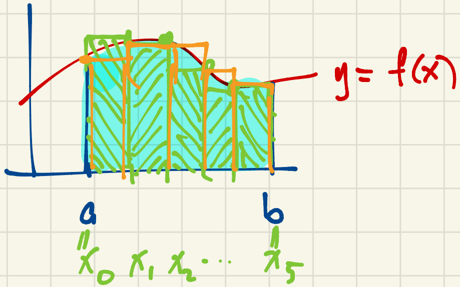
Also: $\lim_{x\to\infty} \dfrac{\sin x}{x} = 0$.

$-\dfrac{1}{x} \le \dfrac{\sin x}{x} \le \dfrac{1}{x}$

0

Sec 5.1

(April 27



Find the area shaded (under the graph of f, above the x-axis, for $a \le x \le b$).

Idea: Take large n. Divide $[a,b]$ into n subintervals each width $\Delta x = \frac{b-a}{n}$. Mark off points $x_0 = a$, x_1, x_2, ..., $x_n = b$ equally spaced Δx apart. i.e. $x_i = a + i\Delta x$.

The Right Riemann Sum with n=5 approximates the shaded area as

$$f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x = \sum_{i=1}^{5} f(x_i)\Delta x$$

(the sum of $f(x_i)\Delta x$ as i ranges from 1 to 5)

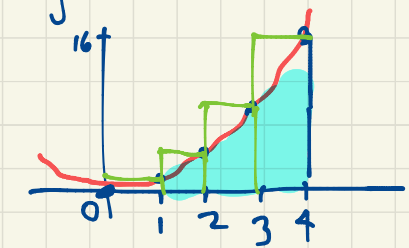
The left Riemann Sum with n=5 subintervals is

$$\sum_{i=1}^{5} f(x_{i-1})\Delta x = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x.$$

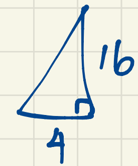
The midpoint Riemann Sum with n=5 subintervals is

$$\sum_{i=1}^{5} f\Big(\underbrace{\frac{x_{i-1}+x_i}{2}}_{\substack{\text{midpoint of} \\ \text{the } i^{th} \text{ subinterval} \\ [x_{i-1},\, x_i]}}\Big)\Delta x = f\left(\frac{x_0+x_1}{2}\right)\Delta x + f\left(\frac{x_1+x_2}{2}\right)\Delta x + \cdots + f\left(\frac{x_4+x_5}{2}\right)\Delta x$$

Eg. We want the area of $0 \le y \le x^2$, $0 \le x \le 4$.



Compare:



triangle has area $\frac{1}{2} \times 4 \times 16 = 32$.

So our area should be less than 32.

The Right Riemann Sum approximation with $n=4$ subintervals:

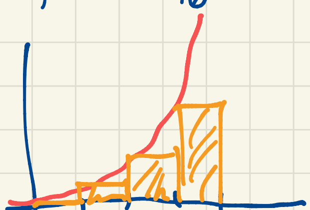
$$\sum_{i=1}^{4} f(x_i)\Delta x = 1\cdot 1 + 4\cdot 1 + 9\cdot 1 + 16\cdot 1 = 30.$$

x	$f(x)$
0	0
1	1
2	4
3	9
4	16

$$\Delta x = \frac{4-0}{4} = 1$$

The left Riemann Sum approximation with $n=4$ subintervals is

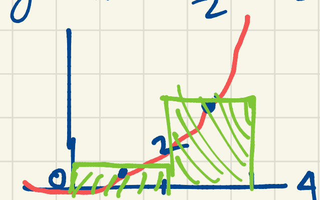
$$\sum_{i=1}^{4} f(x_{i-1})\Delta x = 0\cdot 1 + 1\cdot 1 + 4\cdot 1 + 9\cdot 1 = 14.$$



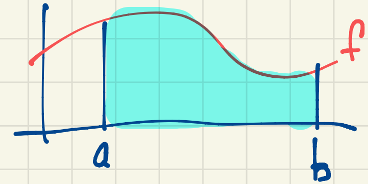
the true area is between 14 and 30. (The average is $\frac{30+14}{2} = 22$.)

The midpoint Riemann Sum with $n=2$ subintervals, $\Delta x = \frac{4-0}{2} = 2$:

$$\sum_{i=1}^{2} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x = f\left(\frac{0+2}{2}\right)2 + f\left(\frac{2+4}{2}\right)2 = f(1)\cdot 2 + f(3)\cdot 2 = 1\cdot 2 + 9\cdot 2 = 20.$$



A Riemann Sum approximation to the area with n subintervals of width $\Delta x = \frac{b-a}{n}$ has the form



$$\sum_{i=1}^{n} f(x_i^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

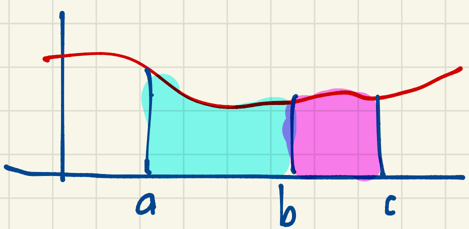
pick a point x_i^* in the i^{th} subinterval $[x_{i-1}, x_i]$

Now let $n \to \infty$.

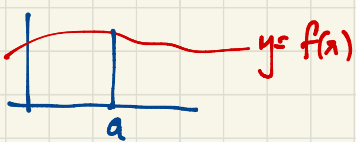
$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\,dx \text{ , the Riemann integral of } f \text{ from } a \text{ to } b.$$

April 28

Sec 5.2 (mostly) Properties of Integrals



$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



$$\int_a^a f(x)\,dx = 0$$

$$\int_a^b f(x)\,dx + \int_b^a f(x)\,dx = \int_a^a f(x)\,dx = 0$$

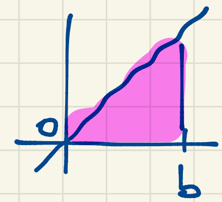
so $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \qquad \Delta x = \frac{b-a}{n}$$

or x_i^*

$$\boxed{\int_0^b x\,dx = \frac{b^2}{2}}$$

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$



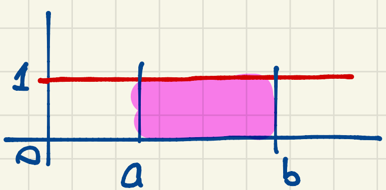
$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

c = constant

$$\int_0^b 1\,dx = b$$

$$\int_a^b 1\,dx = \int_a^b dx = b-a$$

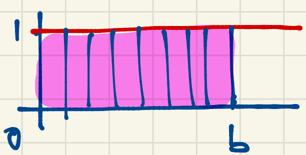
$$\int_0^b x\,dx = \frac{b^2}{2}$$



$$\underbrace{\int_0^a dx}_{a} + \underbrace{\int_a^b dx}_{b-a} = \underbrace{\int_0^b dx}_{b}$$

$$\int_0^b x^2\,dx = \frac{b^3}{3}$$

$$\int_0^b x^k\,dx = \frac{b^{k+1}}{k+1}$$



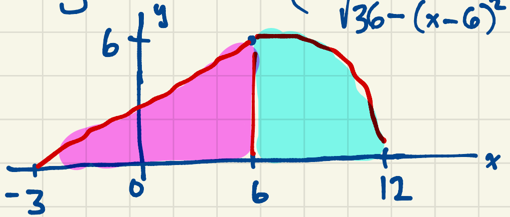
$$\int_0^b dx = b$$

if $k \neq -1$

Compare:

$$\int x^k\,dx = \frac{x^{k+1}}{k+1} \text{ if } k \neq -1$$

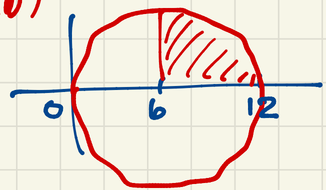
Eg. $f(x) = \begin{cases} \frac{3}{2}(x+3), & \text{for } -3 \le x \le 6 \\ \sqrt{36-(x-6)^2} \end{cases}$



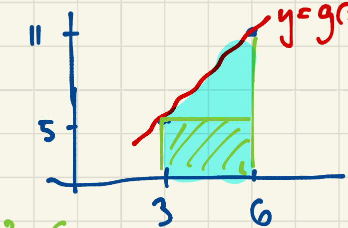
$$\int_{-3}^{12} f(x)\,dx = \tfrac{1}{2}\times 9\times 6 + \frac{36\pi}{4} = 27 + 9\pi$$

circle
center at (6,0)
radius 6

$(x-6)^2 + y^2 = 36 = 6^2$



$$\int_3^6 g(x)\,dx$$



$= 3\times 5 + \tfrac{1}{2}\times 3\times 6$

$= 15 + 9 = 24.$

Second way: $\int_3^6 g(x)\,dx = \underbrace{3}_{\text{base}} \times \underbrace{\frac{5+11}{2}}_{\text{average height}} = 3\times 8 = 24.$

Third way: $y=g(x)$ has slope $\frac{11-5}{6-3} = \frac{6}{3} = 2$

Its equation is $y-5 = 2(x-3)$

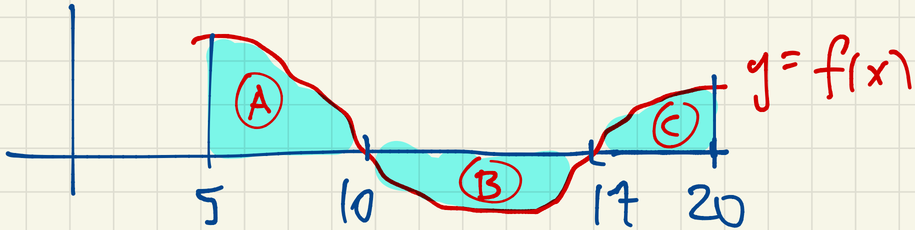
i.e. $y = 2x-1$

$$\int_0^6 g(x)\,dx = \int_0^6 (2x-1)\,dx = 2\int_0^6 x\,dx - \int_0^6 dx$$

$$= 2\cdot\frac{6^2}{2} - 6 = 30$$

$$\int_3^6 g(x)\,dx = \int_0^6 g(x)\,dx - \int_0^3 g(x)\,dx \qquad \int_0^3 g(x)\,dx = 2\cdot\frac{3^2}{2} - 3 = 6$$

$= 30 - 6 = 24.$



$$\int_5^{20} f(x)\,dx = \int_5^{10} f(x)\,dx + \int_{10}^{17} f(x)\,dx + \int_{17}^{20} f(x)\,dx$$

area of (A)

− (area of (B))

area of (C)

Eg. $\displaystyle\int_{-5}^{5} \frac{x}{x^{10}+1}\,dx = 0$ by symmetry