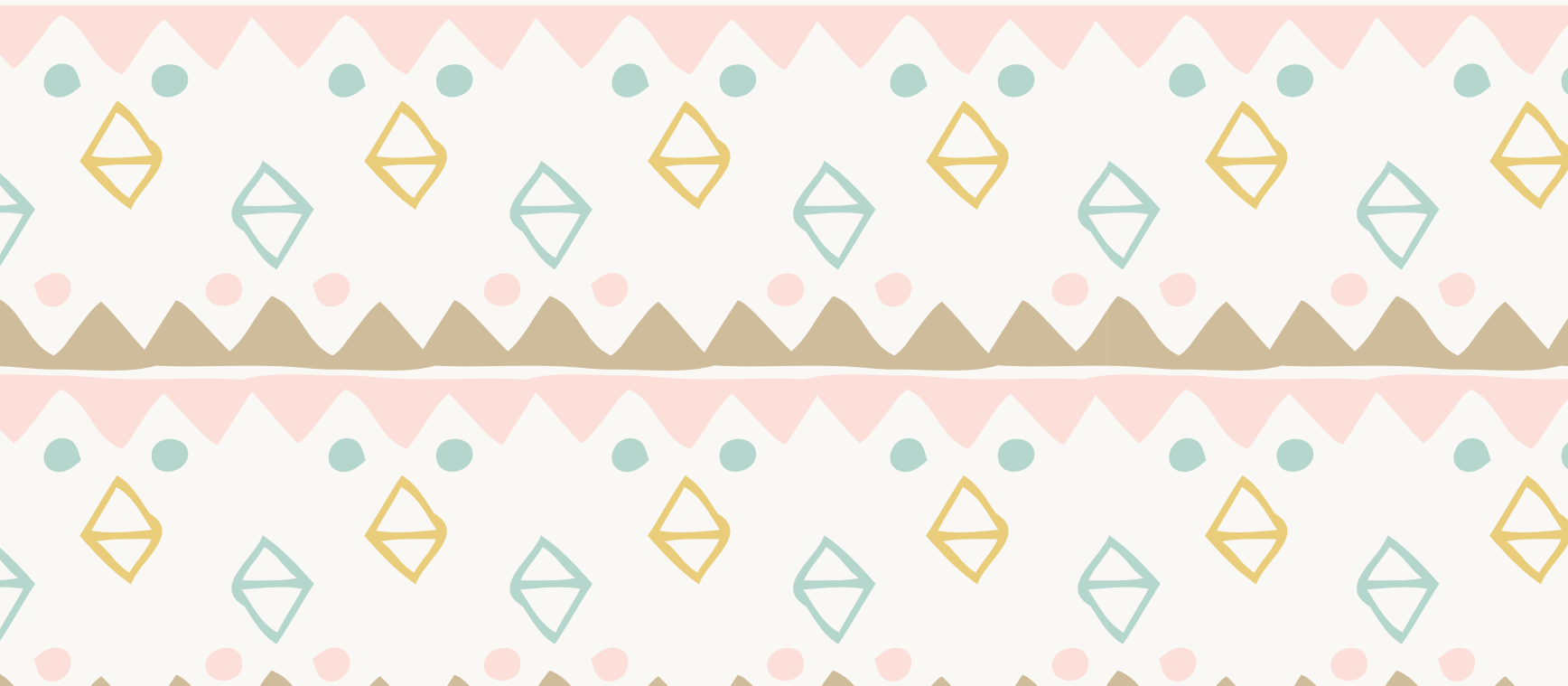


# Math 2200-01 (Calculus I) Spring 2020

Book 3



# Sec 4.5: Optimization

April 13

p.285 #19. of all boxes with a square base and a volume  $8\text{m}^3$ , which one has the minimum surface area?



volume  $V = x^2 h = 8 \Rightarrow h = \frac{8}{x^2}$

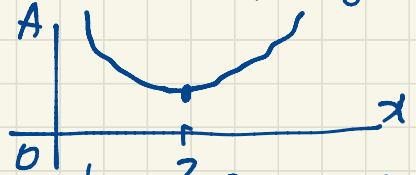
area  $A = \underbrace{2x^2}_{\text{top and bottom}} + \underbrace{4xh}_{\text{sides}} = 2x^2 + 4x \cdot \frac{8}{x^2} = 2x^2 + \frac{32}{x}, x > 0$

The domain is  $(0, \infty)$ , an unbounded open interval.

$$\frac{dA}{dx} = 4x - \frac{32}{x^2} = \frac{1}{x^2} (x^3 - 8)$$

The critical point is at  $x=2$ .  
 (Recall: critical points are where the derivative is zero or undefined. There are no points of the domain where  $\frac{dA}{dx}$  is undefined, only one point where  $\frac{dA}{dx} = 0$ .)

For  $0 < x < 2$ ,  $\frac{dA}{dx} < 0$  so  $A(x)$  is decreasing.  
 For  $x=2$ ,  $\frac{dA}{dx} = 0$ .  
 For  $x > 2$ ,  $\frac{dA}{dx} > 0$  so  $A(x)$  is increasing.

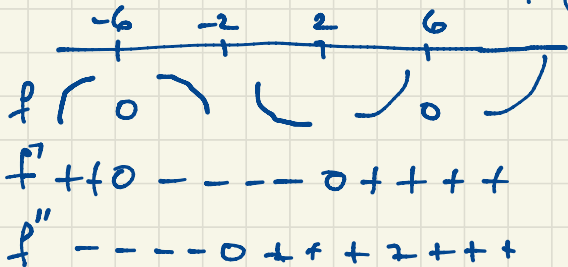


So the minimum surface area  $A(2) = 12\text{m}^2$  occurs for a box of size  $2\text{m} \times 2\text{m} \times 2\text{m}$ .

Sec 4.4. p. 278 # 21.  $f(x) = (x-6)(x+6)^2 = (x^2-36)(x+6) = x^3 + 6x^2 - 36x - 216$

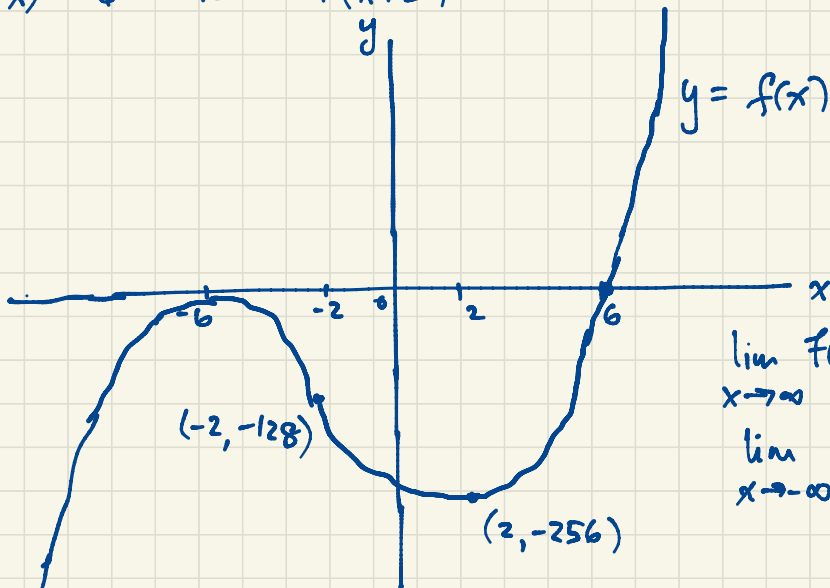
$$f'(x) = 3x^2 + 12x - 36 = 3(x^2 + 4x - 12) = 3(x+6)(x-2)$$

$$f''(x) = 6x + 12 = 6(x+2)$$



$$f(-2) = (-8)(4)^2 = -128$$

$$f(2) = (-4)(8^2) = -256$$



$$\lim_{x \rightarrow \infty} f(x) = \infty$$

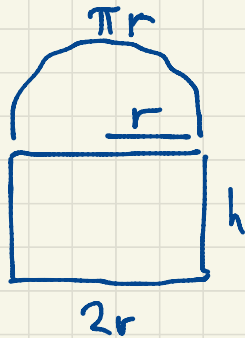
$$\lim_{x \rightarrow -\infty} f(x) = -\infty$$

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$f$  is increasing on  $(-\infty, -6)$  and on  $(2, \infty)$ ,  
 decreasing on  $(-6, 2)$ ,  
 concave down on  $(-\infty, -2)$ ,  
 concave up on  $(-2, \infty)$ .

$f$  has an inflection point  $(-2, -128)$ ,  
 a local minimum point  $(2, -256)$ ,  
 a local maximum point  $(-6, 0)$ ,  
 no absolute extrema or asymptotes.

Sec 4.5 p. 287 #41. Let  $r$  be the radius of the semicircular window pane.



The perimeter is  $P = \pi r + 2r + 2h = 20$

$$(\pi + 2)r + 2h = 20$$

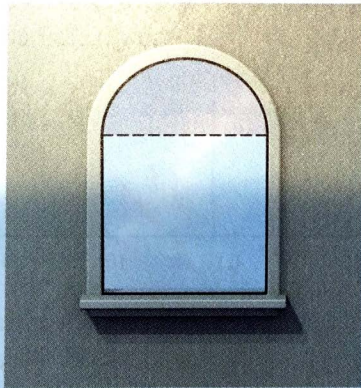
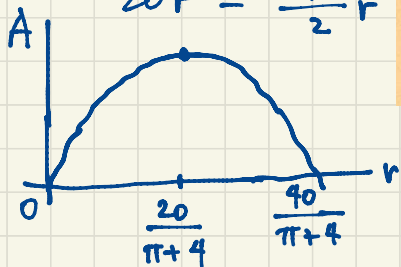
$$2h = 20 - \pi r - 2r$$

$$h = 10 - \frac{\pi + 2}{2}r$$

41. A window consists of rectangular pane of glass surmounted by a semicircular pane of glass (see figure). If the perimeter of the window is 20 feet, determine the dimensions of the window that maximize the area of the window.

$$\frac{dA}{dr} = 20 - (\pi + 4)r$$

$$\begin{aligned} A &= \frac{\pi}{2}r^2 + 2rh \\ &= \frac{\pi}{2}r^2 + 2r\left(10 - \frac{\pi + 2}{2}r\right) \\ &= 20r + \left(\frac{\pi}{2} - (\pi + 2)\right)r^2 \\ &= 20r - \frac{\pi + 4}{2}r^2 \end{aligned}$$



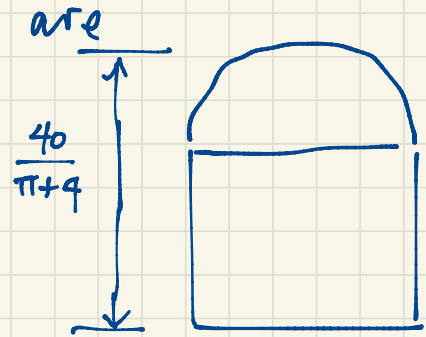
The critical point is at  $r = \frac{20}{\pi + 4}$ .  
When  $0 < r < \frac{20}{\pi + 4}$ ,  $\frac{dA}{dr} > 0$  so  $A$  is increasing.

When  $\frac{20}{\pi + 4} < r < \frac{40}{\pi + 4}$ ,  $\frac{dA}{dr} < 0$  so  $A$  is decreasing.

$$A = \left(20 - \frac{\pi + 4}{2}r\right)r$$

So the maximum area occurs when  $r = \frac{20}{\pi + 4}$ . Alternatively since  $A \geq 0$  requires  $r$  to be in  $\left[0, \frac{40}{\pi + 4}\right]$ , we need only check  $A$  at endpoints and the critical point.

The dimensions of the window that maximizes the area



$$r = \frac{20}{\pi+4}$$

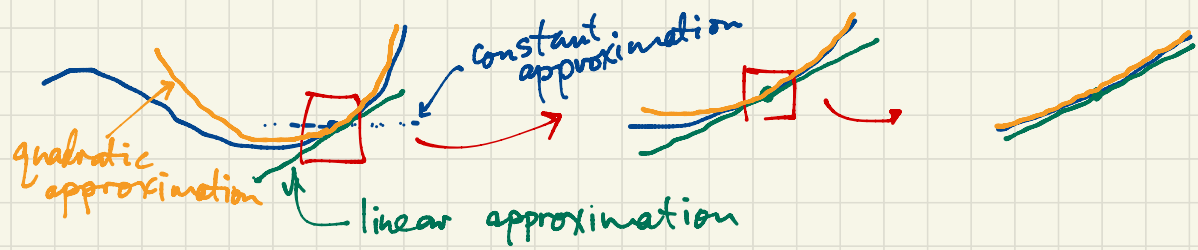
$$h = 10 - \frac{\pi+2}{2}r = 10 - \frac{\pi+2}{2} \cdot \frac{20}{\pi+4}$$

$$= 10 - \frac{10(\pi+2)}{\pi+4}$$

$$= \frac{10(\pi+4) - 10(\pi+2)}{\pi+4}$$

$$2r = \frac{40}{\pi+4}$$

Sec 4.6. Linearization and Differentials  $= \frac{20}{\pi+4}$



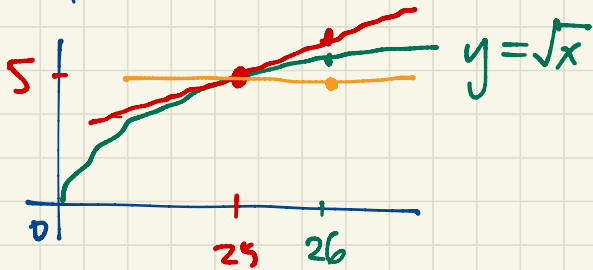
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For  $x$  close to  $a$ ,  $f(x) \approx L(x) = f'(a)(x-a) + f(a) = \boxed{\phantom{00}}x + \boxed{\phantom{00}}$

approximately equal to  $\underbrace{\hspace{10em}}_{\text{linearization of } f \text{ at } (a, f(a))}$

$f'(a)$   $f(a) - af'(a)$

Example: Use the linearization of  $\sqrt{x}$  at  $(25, 5)$  to approximate  $\sqrt{26}$ .



$$f(x) = \sqrt{x}, \quad f(25) = 5$$

$$f'(x) = \frac{1}{2\sqrt{x}}, \quad f'(25) = \frac{1}{2\sqrt{25}} = \frac{1}{10}$$

The linearization of  $\sqrt{x}$  at  $(25, 5)$  is

$$\begin{aligned} L(x) &= f'(25)(x-25) + f(25) \\ &= \frac{1}{10}(x-25) + 5. \end{aligned}$$

If  $x \approx 25$  then  $\sqrt{x} \approx \frac{1}{10}(x-25) + 5$ .

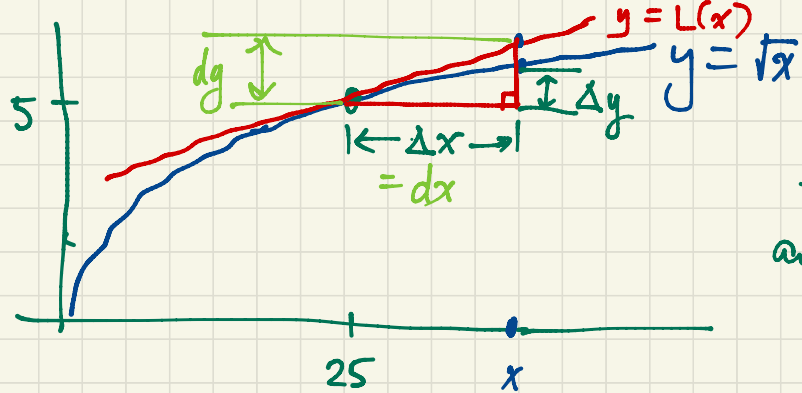
$$\text{Eq. } \sqrt{26} \approx \frac{1}{10}(26-25) + 5 = 5.1$$

$$\text{Check: } \sqrt{26} \approx 5.099019514$$

correct to 3 significant digits.

Coarser approximation:  $\sqrt{26} \approx 5$ . (correct to one significant digit)

Constant approximation  $\sqrt{x} \approx 5$



If we move from  $(25, 5)$  to  $(x, f(x))$  on the graph, our actual function  $f(x) = \sqrt{x}$  changes by an exact amount

$$\Delta g = f(x) - f(25).$$

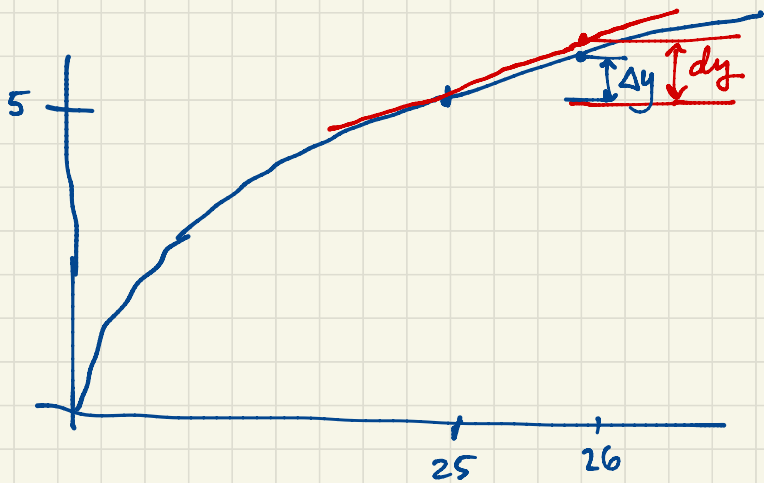
The secant line from  $(25, 5)$  to  $(x, f(x))$  has slope  $\frac{\Delta g}{\Delta x} = \frac{f(x) - f(25)}{x - 25}$ .

On the tangent line,

$$\frac{\Delta L(x)}{\Delta x} = \frac{L(x) - L(25)}{x - 25} = f'(25) = \frac{dy}{dx}$$

Until now we have written  $\frac{dy}{dx}$  as an indivisible symbol. Now we are interpreting  $dx$  and  $dy$  as changes in  $x$  and  $y$ . They are changes on the tangent line (just like  $\Delta x$  and  $\Delta y$  are corresponding changes on the actual curve of  $f$ ).  $dx$  and  $dy$  are differentials.

We'll interpret  $\sqrt{26} \approx 5.1$  in this new language:



$$f'(25) = \frac{1}{10} = \frac{dy}{dx} \Rightarrow dy = \frac{1}{10} dx = \frac{1}{10} \times 1 = 0.1$$

$$dx = \Delta x = 26 - 25 = 1$$

As we move from  $x=25$  to  $x=26$ ,  
the corresponding change in  $y$  is

$$\Delta y \approx dy = 0.1.$$

$$\text{So } \sqrt{26} \approx 5.1$$

This is a quick and easy interpretation for differentials. We will be using differentials throughout Calculus.

Integrals  $\int_a^b f(x) dx$

$$x \rightarrow u \rightarrow y$$

$$\frac{dy}{dx} = \frac{dy}{\cancel{du}} \cdot \frac{\cancel{du}}{dx}$$

If  $y = \sin x$ , find  $\frac{dy}{dx}$  and  $dy$ .

$$\frac{dy}{dx} = \cos x \quad \text{so} \quad dy = (\cos x) dx$$

$$\text{so } \Delta y \approx dy = \cos x dx$$

## Sec 1.7 l'Hôpital's Rule

The limit  $\lim_{x \rightarrow \infty} \frac{x^3 + 2x^2 - x - 1}{2x^3 - 5x} = \lim_{x \rightarrow \infty} \frac{1 + \frac{2}{x} - \frac{1}{x^2} - \frac{1}{x^3}}{2 - \frac{5}{x^2}} = \frac{1}{2}$ .

"indeterminate form  $\frac{\infty}{\infty}$ "

determinate form

The limit  $\lim_{x \rightarrow \infty} \frac{1}{3x^2 + 5} = 0$ .

"determinate form  $\frac{1}{\infty}$ "

Do not confuse l'Hôpital's Rule with the Quotient Rule!

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

indeterminate form  $\frac{0}{0}$

l'Hôpital's Rule For limits of indeterminate form  $\frac{0}{0}$  or  $\frac{\pm\infty}{\pm\infty}$ ,  $\lim_x \frac{f(x)}{g(x)} = \lim_x \frac{f'(x)}{g'(x)}$

assuming the second limit exists.

eg.  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \frac{\cos x}{1} = 1$ .

eg.  $\lim_{x \rightarrow \infty} \frac{x^3 + 2x^2 - x - 1}{2x^3 - 5x} = \lim_{x \rightarrow \infty} \frac{3x^2 + 4x - 1}{6x^2 - 5} = \lim_{x \rightarrow \infty} \frac{6x + 4}{12x} = \lim_{x \rightarrow \infty} \frac{6}{12} = \frac{1}{2}$

$$\text{eg. } \lim_{x \rightarrow \infty} x e^{-x} = \lim_{x \rightarrow \infty} \frac{x}{e^x} = \lim_{x \rightarrow \infty} \frac{1}{e^x} = 0$$

indeterminate form  $\infty \cdot 0$       indeterminate form  $\frac{\infty}{\infty}$

$$\text{eg. } \lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0^+} (-x) = 0.$$

indeterminate form  $0 \cdot (-\infty)$       indeterminate form  $\frac{-\infty}{\infty}$

Don't keep using l'Hôpital's Rule beyond this point; that approach never reaches a conclusion.

$$\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{x}{\frac{1}{\ln x}} = \lim_{x \rightarrow 0^+} \frac{1}{-\frac{1}{(\ln x)^2} \cdot \frac{1}{x}} = \lim_{x \rightarrow 0^+} (-x (\ln x)^2)$$

indeterminate form  $\frac{0}{0}$

this is not an improvement!

$$(a-b)(a+b) = a^2 - b^2$$

$$\lim_{x \rightarrow \infty} \left( \sqrt{x^2 + 10x} - x \right) \cdot \frac{\sqrt{x^2 + 10x} + x}{\sqrt{x^2 + 10x} + x} = \lim_{x \rightarrow \infty} \frac{x^2 + 10x - x^2}{\sqrt{x^2 + 10x} + x} = \lim_{x \rightarrow \infty} \frac{10x}{\sqrt{x^2 + 10x} + x}$$

indeterminate form  $\infty - \infty$

$$= \lim_{x \rightarrow \infty} \frac{10}{\sqrt{1 + \frac{10}{x}} + 1} = \frac{10}{\sqrt{1+0} + 1} = 5.$$

Note:

$$\sqrt{x^2 + 10x} = \sqrt{x^2 \left(1 + \frac{10}{x}\right)} = x \sqrt{1 + \frac{10}{x}}$$

$$\sqrt{x^2} = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

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Another l'Hôpital's Rule example for the indeterminate form  $1^\infty$ .

$$\lim_{x \rightarrow \infty} \left(1 + \frac{a}{x}\right)^x \text{ where } a \text{ is constant.}$$

In order to use l'Hôpital's Rule, we need to rewrite this as a quotient.

$$b^c = (e^{\ln b})^c = e^{c \ln b}$$

$$\lim_{x \rightarrow \infty} \left(1 + \frac{a}{x}\right)^x = \lim_{x \rightarrow \infty} e^{x \ln \left(1 + \frac{a}{x}\right)} = \lim_{u \rightarrow 1} e^{\frac{a}{u-1} \ln u} = e^{a \cdot 1} = e^a.$$

$$u = 1 + \frac{a}{x} \rightarrow 1.$$

$$\lim_{u \rightarrow 1} \frac{\ln u}{u-1} = \lim_{u \rightarrow 1} \frac{1/u}{1} = 1.$$

$$u-1 = \frac{a}{x}$$

$$x = \frac{a}{u-1}$$

(indeterminate form  $\frac{0}{0}$ )

$$\text{If } f(x) = e^{ax} \text{ then } \lim_{u \rightarrow 1} f\left(\frac{\ln u}{u-1}\right) = f\left(\lim_{u \rightarrow 1} \frac{\ln u}{u-1}\right) = f(1) = e^{a \cdot 1} = e^a.$$

Does  $\lim_u f(g(u)) = f(\lim_u g(u))$ ? Can you move the limit inside?

We can do this when  $f$  is continuous and  $g$  is continuous.

$$c = \lim_u g(u), \text{ write } t = g(u) \rightarrow c. \quad \lim_{t \rightarrow c} f(t) = f(c).$$

Some books define  $e^r = \lim_{x \rightarrow \infty} \left(1 + \frac{r}{x}\right)^x$ .

This limit comes from compound interest.

If you deposit a principal amount  $A$  in the bank at nominal interest rate  $r$  (eg. 5% interest per annum would give  $r = 0.05$ ). If interest is compounded annually then after one year you have earned  $rA$  interest. The total balance in the bank after a year would be  $A + rA = (1+r)A$ .

If interest is compounded semiannually (every 6 months) then after 6 months you have  $(1 + \frac{r}{2})A$  as balance after 6 months; then at the end of the year you have  $(1 + \frac{r}{2}) \cdot (1 + \frac{r}{2})A = (1 + \frac{r}{2})^2 A = (1 + r + \frac{r^2}{4})A$ .

If interest is compounded  $n$  times per year then every  $\frac{1}{n}$  year your balance is multiplied by  $1 + \frac{r}{n}$ . After one year your balance is  $(1 + \frac{r}{n})^n A$ .

For continually compounded interest we let  $n \rightarrow \infty$ .

$\lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^n A = e^r A$ . Eg. 5% interest compounded continuously results in

balance  $e^{0.05} A \approx 1.05127 A$  (effectively 5.127% interest rate per annum).

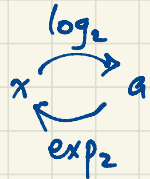
p. 31 # 69.  $\lim_{x \rightarrow \infty} \frac{\log_2 x}{\log_3 x} = \lim_{x \rightarrow \infty} \frac{(\ln x) / \ln 2}{(\ln x) / \ln 3} = \frac{\ln 3}{\ln 2} = \log_2 3$

$\log_2 x = \frac{\ln x}{\ln 2}$  why?

$a = \log_2 x \iff 2^a = x \iff \ln(2^a) = \ln x$

$\iff a \ln 2 = \ln x$

$\iff a = \frac{\ln x}{\ln 2}$



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### Sec 4.6 Linearization and Differentials

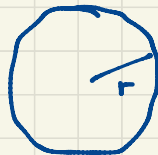
$y = f(x)$

$\frac{dy}{dx} = f'(x)$

$\Delta y \approx dy = f'(x) dx = f'(x) \Delta x$

small change in  $y$   
due to change  
in  $x$ .

Formulas for area and circumference of a circle

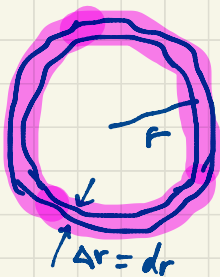


$A = \pi r^2$

$C = 2\pi r$

From the formula  $A = \pi r^2$  you can deduce the formula  $C = 2\pi r$  using linearization.

Starting with a circle of radius  $r$ , area  $A = \pi r^2$ , enlarge the circle by expanding the radius by a small amount  $\Delta r$ .



$$\Delta A \approx C \Delta r \approx C \Delta r$$

$$\Delta A \approx dA = 2\pi r dr = 2\pi r \Delta r$$

$$A = \pi r^2$$

$$\frac{dA}{dr} = 2\pi r$$

$$dA = 2\pi r dr$$

$$\text{So } C = 2\pi r.$$

For a sphere of radius  $r$ ,

$$V = \frac{4}{3}\pi r^3. \quad \text{What is the surface area?}$$

Imagine painting the surface of a ball of radius  $r$ . How much paint is required to paint the surface? If I add a layer of paint of thickness  $\Delta r$  then the new volume changes by

$$\Delta V \approx A \Delta r$$

$$\Delta V \approx dV = 4\pi r^2 dr = 4\pi r^2 \Delta r$$

$$\text{So } A = 4\pi r^2.$$

Sec 4.6 p. 299 #28.  $f(x) = \frac{x}{x+1}$ . Approximate  $f(1.1)$  using the linearization at  $x=1$ .

$f(1) = \frac{1}{2} = 0.5$ . This value is the base for our linearization.

$$L(x) = f(a) + f'(a)(x-a), \quad a=1$$
$$= \frac{1}{2} + \frac{1}{4}(x-1)$$

$$f(1.1) \approx L(1.1) = \frac{1}{2} + \frac{1}{4}(0.1) = 0.5 + 0.025 = 0.525$$

$$\frac{dy}{dx} = f'(x) = \frac{(x+1) \cdot 1 - x \cdot 1}{(x+1)^2}$$

$$= \frac{1}{(x+1)^2}$$

$$dy = \frac{dx}{(x+1)^2}$$

Alternatively using differentials:

$x$  goes from 1 to 1.1, a small change of  $\Delta x = dx = 0.1$ . What is the corresponding change in  $y$ ?

$$\Delta y \approx dy = \frac{dx}{(x+1)^2}$$

$$\Delta y \approx \frac{\Delta x}{(1+1)^2} = \frac{\Delta x}{4} = \frac{0.1}{4} = 0.025$$

$$y \approx 0.5 + \Delta y = 0.525.$$

$$f(1.1) = \frac{1.1}{2.1} = 0.52380952\dots$$

The error is  $0.525 - f(1.1) = 0.00119\dots$

The relative error is  $\frac{0.00119}{0.5238} = 0.0023\dots$

Actually  $0.0023 = 0.23\%$

"per cent" means "divided by 100"

(an over estimate since this difference is positive).

The percentage error is  $0.0023 \times 100\% = 0.23\%$   
(about a quarter of a percent error).

## Sec 4.9 Antiderivatives

$$\frac{d}{dx} (7x^2 + 2x + 1)^3 = 3(7x^2 + 2x + 1)^2 (14x + 2).$$

Reverse: ~~The~~ <sup>An</sup> antiderivative of  $3(7x^2 + 2x + 1)^2 (14x + 2)$  is  $(7x^2 + 2x + 1)^3$ .

The general answer for antiderivative:  $(7x^2 + 2x + 1)^3 + \text{constant}$ .

$$\frac{d}{dx} [(7x^2 + 2x + 1)^3 + 11] = 3(7x^2 + 2x + 1)^2 (14x + 2).$$

We write  $\int 3(7x^2 + 2x + 1)^2 (14x + 2) dx = (7x^2 + 2x + 1)^3 + C$  where  $C$  is an arbitrary

The general antiderivative of  $3(7x^2 + 2x + 1)^2 (14x + 2)$  with constant respect to  $x$  is  $(7x^2 + 2x + 1)^3 + \text{constant}$ .

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$$\text{eg. } \int x^k dx = \begin{cases} \frac{1}{k+1} x^{k+1}, & \text{if } k \neq -1 \\ \ln|x|, & \text{if } k = -1 \end{cases}$$

$$\text{Check: } \frac{d}{dx} \frac{x^{k+1}}{k+1} = \frac{(k+1)}{k+1} x^k = x^k \quad (k \text{ constant}).$$

$$\frac{d}{dx} (\ln|x|) = x^{-1} = \frac{1}{x}$$

$$\int (x^3 - 7x^2 - 5x + 3) dx = \frac{1}{4}x^4 - \frac{7}{3}x^3 - \frac{5}{2}x^2 + 3x + C$$

Check:  $\frac{d}{dx} \left( \frac{1}{4}x^4 - \frac{7}{3}x^3 - \frac{5}{2}x^2 + 3x + C \right) = x^3 - 7x^2 - 5x + 3$

Compare notation:  $\int f(x) dx$  antiderivative (function) (indefinite integral)

vs.  $\int_a^b f(x) dx$  integral (a number) (definite integral)

↑  
Integrand

How do we find antiderivatives? Best method: guess and check.

Eq.  $\int x \sin x dx = \sin x - x \cos x + C$

$$\frac{d}{dx} (?) = x \sin x$$

$$\frac{d}{dx} (x \cos x) = 1 \cdot \cos x + x \cdot (-\sin x) = \cos x - x \sin x$$

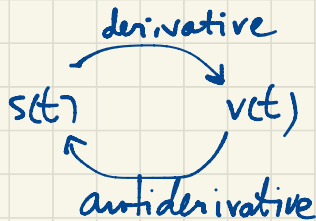
$$\frac{d}{dx} (-\cos x) = \sin x$$

$$\frac{d}{dx} (x \cos x - \sin x) = (\cancel{\cos x} - x \sin x) - \cancel{\cos x} = -x \sin x$$

$$\frac{d}{dx} (\sin x - x \cos x) = x \sin x$$

Sec 4.9 #92. Given velocity  $v(t) = e^t + 4$  } initial value problem  
 and initial position  $s(0) = 2$  }  
 find  $s(t)$ , the position at time  $t$ .

Recall  $s'(t) = v(t)$  so  $s(t) = \int v(t) dt = \int (e^t + 4) dt = e^t + 4t + C$



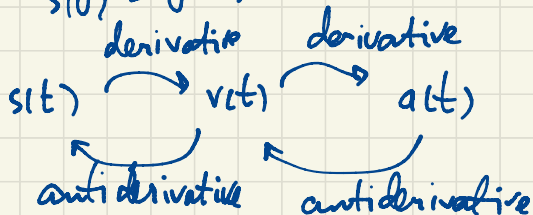
Using  $s(0) = 2$ , find  $C$ .

$$s(t) = e^t + 4t + C$$

$$2 = s(0) = 1 + 0 + C \Rightarrow C = 1$$

So  $s(t) = e^t + 4t + 1$ .

#97.  $a(t) = -32$  } differential equation } initial value problem  
 $v(0) = 20$  } initial conditions }  
 $s(0) = 0$  }



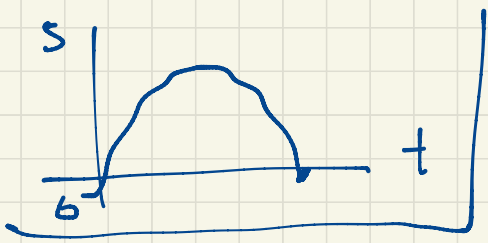
$$v(t) = \int a(t) dt = \int (-32) dt = -32t + C$$

$$20 = v(0) = C \Rightarrow v(t) = -32t + 20$$

$$s(t) = \int v(t) dt = \int (-32t + 20) dt = -16t^2 + 20t + K$$

Since  $s(0) = 0 = K$ ,

$s(t) = -16t^2 + 20t$ .



#100.  $a(t) = 2 \cos t$

$$v(0) = 1$$

$$s(0) = 0$$

$$v(t) = \int a(t) dt = \int 2 \cos t dt = 2 \sin t + C$$

$$1 = v(0) = C \Rightarrow v(t) = 2 \sin t + 1$$

$$s(t) = \int v(t) dt = \int (2 \sin t + 1) dt$$

$$= -2 \cos t + t + K$$

$$0 = s(0) = -2 + K \Rightarrow K = 2$$

$$s(t) = -2 \cos t + t + 2$$

$$\int \frac{dt}{t^2+1} = \int \frac{1}{t^2+1} dt = \tan^{-1} t + C$$

~~$$\int \tan^{-1} t dt = \frac{1}{t^2+1}$$~~

$$\int \frac{dt}{t} = \int \frac{1}{t} dt = \ln |t| + C$$

$$\frac{d}{dt} \frac{1}{t^2+1} = -\frac{1}{(t^2+1)^2} \cdot 2t = -\frac{2t}{(t^2+1)^2}$$

$$\int dt = \int 1 dt = t + C$$

April 24

# Bouncing up on l'Hôpital's Rule

$$\lim_{x \rightarrow \infty} \frac{x^n}{e^x} = 0 \quad (n \text{ applications of l'Hôpital's Rule})$$

$$\text{Eq for } n=4, \quad \lim_{x \rightarrow \infty} \frac{x^4}{e^x} = \lim_{x \rightarrow \infty} \frac{4x^3}{e^x} = \lim_{x \rightarrow \infty} \frac{12x^2}{e^x} = \lim_{x \rightarrow \infty} \frac{24x}{e^x} = \lim_{x \rightarrow \infty} \frac{24}{e^x} = 0.$$

$$\lim_{x \rightarrow \infty} \frac{x^4}{e^{x^2}} = \lim_{x \rightarrow \infty} \frac{4x^3}{2xe^{x^2}} = \lim_{x \rightarrow \infty} \frac{2x^2}{e^{x^2}} = \dots = 0.$$

$$e^{x^2} = e^{(x^2)} \quad \text{or } \cancel{(e^x)^2}$$

$$(e^x)^2 = e^{2x}$$

$$\lim_{x \rightarrow \infty} \frac{x^4}{e^{x^2}} = \lim_{u \rightarrow \infty} \frac{u^2}{e^u} = 0$$

$$u = x^2$$

$$\text{OR: } 0 < \frac{x^4}{e^{x^2}} < \frac{x^4}{e^x} \quad \lim_{x \rightarrow \infty} \frac{x^4}{e^{x^2}} = 0 \text{ by the Squeeze Theorem.}$$

$$\text{p. 311 \# 28. } \lim_{x \rightarrow 0^+} \frac{x - 3\sqrt{x}}{x - \sqrt{x}} = \lim_{x \rightarrow 0^+} \frac{1 - \frac{3}{2\sqrt{x}}}{1 - \frac{1}{2\sqrt{x}}} = \lim_{x \rightarrow 0^+} \frac{2\sqrt{x} - 3}{2\sqrt{x} - 1} = 3.$$

$$\text{OR } \lim_{x \rightarrow 0^+} \frac{x - 3\sqrt{x}}{x - \sqrt{x}} = \lim_{x \rightarrow 0^+} \frac{\sqrt{x} - 3}{\sqrt{x} - 1} = \frac{0 - 3}{0 - 1} = 3.$$

$$\#40. \lim_{x \rightarrow 0} \frac{\sin x - x}{7x^3} = \lim_{x \rightarrow 0} \frac{\cos x - 1}{21x^2} = \lim_{x \rightarrow 0} \frac{-\sin x}{42x} = \lim_{x \rightarrow 0} \frac{-\cos x}{42} = -\frac{1}{42}$$

$$\#83. \lim_{x \rightarrow 0} (x + \cos x)^{1/x} = \lim_{x \rightarrow 0} e^{\frac{1}{x} \ln(x + \cos x)} = e' = e$$

indeterminate form  $1^\infty$

using  $e^{\ln a} = a$ ,  $e^{\ln(x + \cos x)} = (x + \cos x)$

Compare:  $\lim_{x \rightarrow 0^+} (x+1)^{1/x} = \lim_{t \rightarrow \infty} \left(1 + \frac{1}{t}\right)^t = e$

$$t = \frac{1}{x} \rightarrow \infty$$

$$\lim_{x \rightarrow 0} \frac{\ln(x + \cos x)}{x} = \lim_{x \rightarrow 0} \frac{\frac{1}{x + \cos x} (1 - \sin x)}{1} = 1$$

indeterminate form  $\frac{0}{0}$

$$\#68. \lim_{x \rightarrow 0^+} x^{\frac{1}{\ln x}} = \lim_{x \rightarrow 0^+} e^{\frac{1}{\ln x} \ln x} = \lim_{x \rightarrow 0^+} e = e$$

$$x = e^{\ln x} \Rightarrow x^{\frac{1}{\ln x}} = (e^{\ln x})^{\frac{1}{\ln x}} = e$$

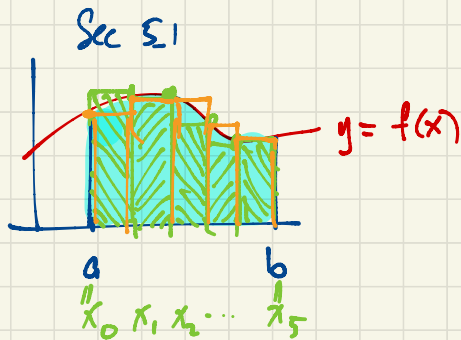
$$\#20. \lim_{x \rightarrow 0} \frac{e^x - 1}{2x + 5} = \frac{0}{5} = 0$$

Also:

$$\lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0$$

$$\frac{-1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$$

April 27



Find the area shaded (under the graph of  $f$ , above the  $x$ -axis, for  $a \leq x \leq b$ ).

Idea: Take large  $n$ . Divide  $[a, b]$  into  $n$  subintervals each width  $\Delta x = \frac{b-a}{n}$ . Mark off points  $x_0 = a, x_1, x_2, \dots, x_n = b$  equally spaced  $\Delta x$  apart. i.e.  $x_i = a + i\Delta x$ .

The Right Riemann Sum <sup>with  $n=5$</sup>  approximates the shaded area as

$$f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x = \sum_{i=1}^5 f(x_i)\Delta x \quad (\text{the sum of } f(x_i)\Delta x \text{ as } i \text{ ranges from 1 to 5})$$

The left Riemann sum with  $n=5$  subintervals is

$$\sum_{i=1}^5 f(x_{i-1})\Delta x = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x.$$

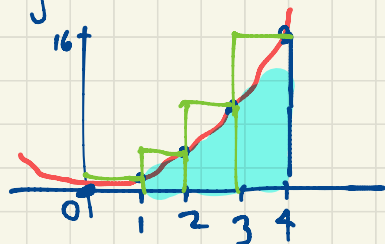
The midpoint Riemann sum with  $n=5$  subintervals is

$$\sum_{i=1}^5 f\left(\frac{x_{i-1} + x_i}{2}\right)\Delta x = f\left(\frac{x_0 + x_1}{2}\right)\Delta x + f\left(\frac{x_1 + x_2}{2}\right)\Delta x + \dots + f\left(\frac{x_4 + x_5}{2}\right)\Delta x$$

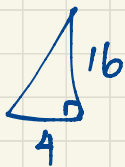
midpoint of  
the  $i^{\text{th}}$  subinterval

$$[x_{i-1}, x_i]$$

Ex. We want the area of  $0 \leq y \leq x^2$ ,  $0 \leq x \leq 4$ .



Compare:



triangle has area  $\frac{1}{2} \times 4 \times 16 = 32$ .

So our area should be less than 32.

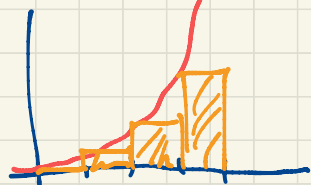
The Right Riemann Sum approximation with  $n=4$  subintervals:

$$\sum_{i=1}^4 f(x_i) \Delta x = 1 \cdot 1 + 4 \cdot 1 + 9 \cdot 1 + 16 \cdot 1 = 30.$$

$$\Delta x = \frac{4-0}{4} = 1$$

The left Riemann Sum approximation with  $n=4$  subintervals is

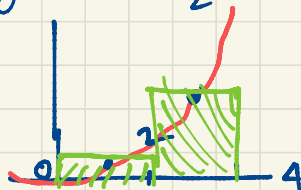
$$\sum_{i=1}^4 f(x_{i-1}) \Delta x = 0 \cdot 1 + 1 \cdot 1 + 4 \cdot 1 + 9 \cdot 1 = 14.$$



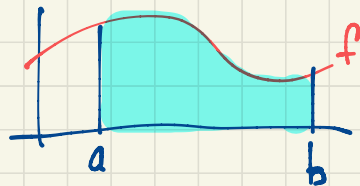
the true area is between 14 and 30. (The average is  $\frac{30+14}{2} = 22$ )

The midpoint Riemann Sum with  $n=2$  subintervals,  $\Delta x = \frac{4-0}{2} = 2$ :

$$\sum_{i=1}^2 f\left(\frac{x_{i-1}+x_i}{2}\right) \Delta x = f\left(\frac{0+2}{2}\right) 2 + f\left(\frac{2+4}{2}\right) 2 = f(1) \cdot 2 + f(3) \cdot 2 = 1 \cdot 2 + 9 \cdot 2 = 20.$$



A Riemann Sum approximation to the area with  $n$  subintervals of width  $\Delta x = \frac{b-a}{n}$  has the form



$$\sum_{i=1}^n f(x_i^*) \Delta x = f(x_1^*) \Delta x + f(x_2^*) \Delta x + \dots + f(x_n^*) \Delta x$$

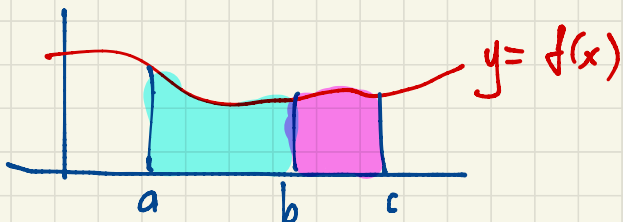
pick a point  $x_i^*$   
in the  $i^{\text{th}}$  subinterval  $[x_{i-1}, x_i]$

Now let  $n \rightarrow \infty$ .

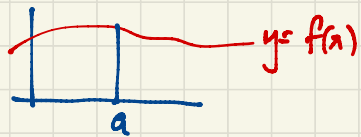
$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x = \int_a^b f(x) dx, \text{ the Riemann integral of } f \text{ from } a \text{ to } b.$$

Sec 5.2 (mostly) Properties of Integrals

April 28



$$\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$$



$$\int_a^a f(x) dx = 0$$

$$\int_a^b f(x) dx + \int_b^a f(x) dx = \int_a^a f(x) dx = 0$$

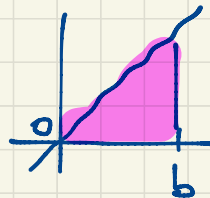
$$\text{so } \int_b^a f(x) dx = - \int_a^b f(x) dx$$

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x \quad \Delta x = \frac{b-a}{n}$$

↑  
or  $x_i^*$

$$\int_0^b x dx = \frac{b^2}{2}$$

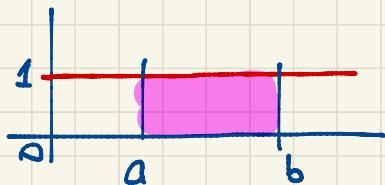
$$\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx$$



$$\int_a^b c f(x) dx = c \int_a^b f(x) dx$$

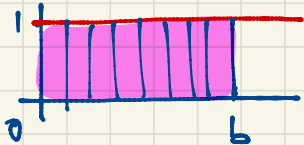
$c = \text{constant}$

$$\int_a^b 1 dx = \int_a^b dx = b - a$$



$$\underbrace{\int_0^a dx}_a + \underbrace{\int_a^b dx}_{b-a} = \underbrace{\int_0^b dx}_b$$

$$\int_0^b dx = b$$



$$\int_0^b 1 dx = b$$

$$\int_0^b x dx = \frac{b^2}{2}$$

$$\int_0^b x^2 dx = \frac{b^3}{3}$$

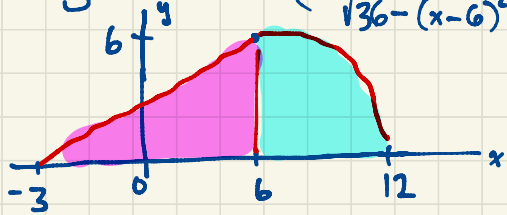
$$\int_0^b x^k dx = \frac{b^{k+1}}{k+1}$$

Compare:

$$\int x^k dx = \frac{x^{k+1}}{k+1} \text{ if } k \neq -1$$

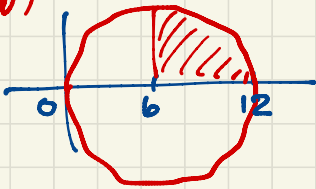
if  $k \neq -1$

Ex.  $f(x) = \begin{cases} \frac{3}{2}(x+3) & \text{for } -3 \leq x \leq 6 \\ \sqrt{36 - (x-6)^2} & \text{for } 6 \leq x \leq 12 \end{cases}$



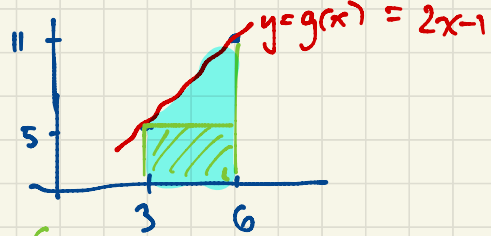
circle  
center at  $(6, 0)$   
radius  $b$

$$(x-6)^2 + y^2 = 36 = 6^2$$



$$\int_{-3}^{12} f(x) dx = \frac{1}{2} \times 9 \times 6 + \frac{36\pi}{4} = 27 + 9\pi$$

$$\int_3^6 g(x) dx = 3 \times 5 + \frac{1}{2} \times 3 \times 6 = 15 + 9 = 24$$



Second way:  $\int_3^6 g(x) dx = 3 \times \frac{5+11}{2} = 3 \times 8 = 24$

base      average height

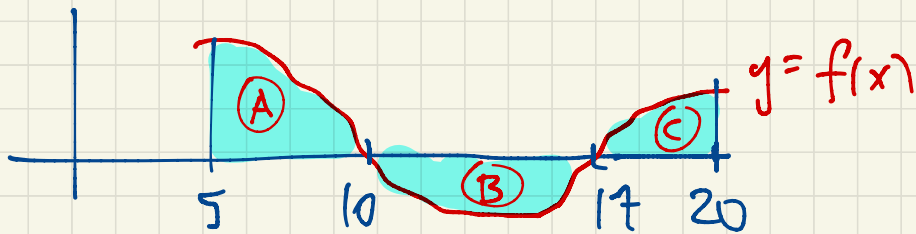
Third way:  $y = g(x)$  has slope  $\frac{11-5}{6-3} = \frac{6}{3} = 2$

Ifs equation is  $y - 5 = 2(x - 3)$   
i.e.  $y = 2x - 1$

$$\int_3^6 g(x) dx = \int_0^6 g(x) dx - \int_0^3 g(x) dx = 30 - 6 = 24$$

$$\int_0^3 g(x) dx = 2 \cdot \frac{3^2}{2} - 3 = 6$$

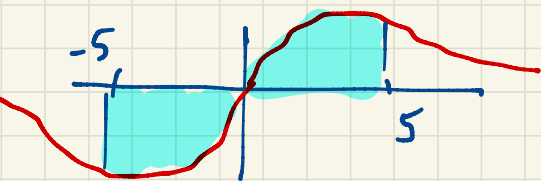
$$\int_0^6 g(x) dx = 2 \cdot \frac{6^2}{2} - 6 = 30$$



$$\int_5^{20} f(x) dx = \int_5^{10} f(x) dx + \int_{10}^{17} -f(x) dx + \int_{17}^{20} f(x) dx$$

area of (A)  $-$  (area of (B))  $+$  area of (C)

Eg.  $\int_{-5}^5 \frac{x}{x^{10}+1} dx = 0$  by symmetry



### Sec 5.2 Fundamental Theorem of Calculus

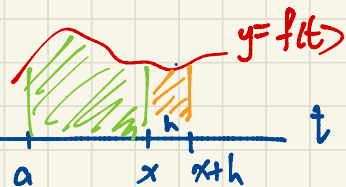
(April 29)

Let  $a, x$  be points in an interval where  $f$  is defined. Look at

$$F(x+h) \approx F(x) + h f(x)$$

$$f(x) \approx \frac{F(x+h) - F(x)}{h}$$

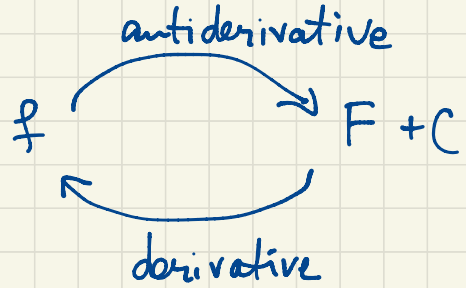
$$f(x) = \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} = F'(x)$$



Let  $F(x) = \int_a^x f(t) dt$ . Then  $F'(x) = f(x)$ .

Why?

$$\int_a^{x+h} f(t) dt = \underbrace{\int_a^x f(t) dt}_{F(x)} + \underbrace{\int_x^{x+h} f(t) dt}_{\approx h f(x)} = F(x) + h f(x)$$



$$F(x) = \int_a^x f(t) dt$$

$$F'(x) = f(x)$$

This is one form of the Fundamental Theorem of Calculus.

$$\frac{d}{dx} \int_a^x f(t) dt = f(x)$$

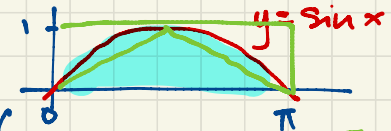
In a nutshell: If you start with  $f$  and you integrate, then differentiate, you get back the original  $f$ .

The other way around gives the other form of the Fundamental Theorem of Calculus: If you start with a function  $F$ , take its derivative, then integrate, you get back  $F + C$ .

eg. Find the shaded area under the first arch of the sine curve:

$$\text{Area} = \int_0^{\pi} f(x) dx = F(\pi) = 1 - \cos \pi = 1 - (-1) = 2$$

$$F(x) = 1 - \cos x$$



$$\text{Use } F(x) = \int_0^x \sin t dt, \quad F'(x) = \sin x \Rightarrow F(x) = -\cos x + C$$

$$0 = F(0) = -\cos 0 + C \Rightarrow C = 1$$

$\frac{\pi}{2} < \text{Area} < \pi$

The mechanics of using the Fundamental Theorem of Calculus to get an integral:  
Evaluate the antiderivative between the two endpoints.

$$\int_0^{\pi} \sin x \, dx = F(\pi) - F(0) = \underbrace{-\cos(\pi)}_{F(\pi)} - \underbrace{(-\cos 0)}_{F(0)} = 1 + 1 = 2.$$

First find an antiderivative  $F(x)$  for  $\sin x$ . I can use  $F(x) = -\cos x$ .  
(Any antiderivative will do because the  $+C$  will cancel in the end.)

In practice we write:

$$\int_0^{\pi} \sin x \, dx = \left[ \cos x \right]_0^{\pi} = -\cos \pi - (-\cos 0) = 1 + 1 = 2.$$

Notation:

$$\left[ F(x) \right]_a^b = F(b) - F(a).$$

Note: If  $F(x) = \int f(x) \, dx$  (antiderivative)

$$\text{then } \left[ F(x) \right]_a^b = F(b) - F(a) = \int_a^b f(x) \, dx$$

$$\text{i.e. } \left[ \int f(x) \, dx \right]_a^b = \int_a^b f(x) \, dx$$

eg.  $\left[ e^{x^2} \right]_1^3 = e^9 - e$

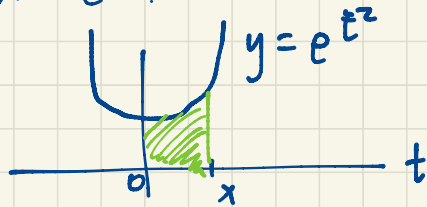
$$\text{Eg. } \int_1^3 (6x^2 + 4x + 3) dx = \left[ 2x^3 + 2x^2 + 3x \right]_1^3 = \left[ 2 \times 27 + 2 \times 9 + 9 \right] - \left[ 2 + 2 + 3 \right] \\ = 81 - 7 = 74.$$

$$\int_1^3 e^{x^2} dx =$$

Find an antiderivative for  $e^{x^2}$ ? No nice formula for it.

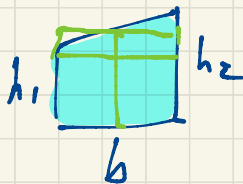
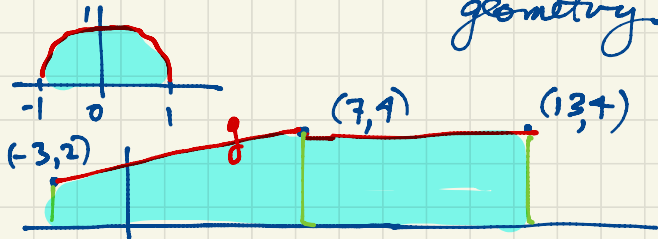
I know there is an antiderivative for  $e^{x^2}$ . Here it is:

$$F(x) = \int_0^x e^{t^2} dt$$



May 1

$$\int_{-1}^1 \sqrt{1-x^2} dx = \frac{\pi}{2} \text{ using high school geometry.}$$

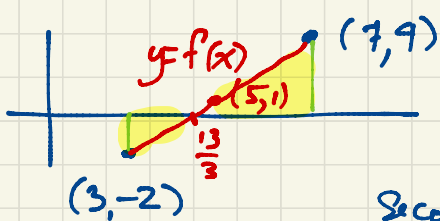


Area of trapezoid is

$$\frac{h_1 + h_2}{2} b$$

$$\frac{4+2}{2} = 3$$

$$\int_{-3}^{13} g(x) dx = 10 \times 3 + 6 \times 4 \\ = 54$$



midpoint Riemann sum with one subinterval

$$\int_3^7 f(x) dx = 4 \times 1 = 4.$$

$$\text{slope} = \frac{6}{4} = \frac{3}{2}$$

$$y + 2 = \frac{3}{2}(x - 3)$$

$$y = \frac{3}{2}x - \frac{13}{2}$$

Second solution:

$$x\text{-intercept is: } \frac{13}{3}$$

$$\text{solve } 0 = \frac{3}{2}x - \frac{13}{2}$$

$$\int_3^7 f(x) dx = (\text{Area above } x\text{-axis}) - (\text{Area below } x\text{-axis})$$

$$= \frac{(7 - \frac{13}{3}) \times 4}{2} - \frac{(\frac{13}{3} - 3) \times 2}{2}$$

$$= \frac{\frac{8}{3} \times 4}{2} - \frac{\frac{4}{3} \times 2}{2} = \frac{16}{3} - \frac{4}{3} = \frac{12}{3} = 4.$$

Third solution: Use the Fundamental Theorem of Calculus

$$\int_3^7 \left(\frac{3}{2}x - \frac{13}{2}\right) dx = \underbrace{\left[\frac{3}{4}x^2 - \frac{13}{2}x\right]}_{F(x)} \Big|_3^7 = \underbrace{\left[\frac{3 \times 49}{4} - \frac{13 \times 7}{2}\right]}_{F(7)} - \underbrace{\left[\frac{3 \times 9}{4} - \frac{13 \times 3}{2}\right]}_{F(3)}$$

$$\begin{aligned} &= \left[\frac{147}{4} - \frac{91}{2}\right] - \left[\frac{27}{4} - \frac{39}{2}\right] \\ &= \frac{120}{4} - \frac{52}{2} = 30 - 26 \\ &= 4. \end{aligned}$$

$$\int_1^3 (6x^2 - 5x + 2) dx = \left[ 2x^3 - \frac{5}{2}x^2 + 2x \right]_1^3 = \left[ 54 - \frac{45}{2} + 6 \right] - \left[ 2 - \frac{5}{2} + 2 \right]$$

$$= 56 - \frac{45}{2} + \frac{5}{2} = 56 - \frac{40}{2} = 56 - 20 = 36.$$

$$\int x^k dx = \frac{x^{k+1}}{k+1} \quad \text{if } k \neq -1$$

$$\int_0^1 x e^x dx = (x-1)e^x \Big|_0^1 = 0 - (-1) = 1.$$

Use the Fundamental Theorem of Calculus:

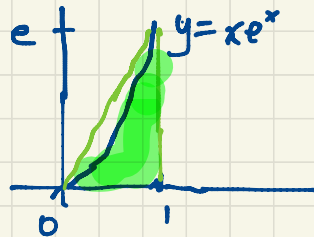
$$\frac{d}{dx} (?) = x e^x ?$$

Guess and check:

$$\frac{d}{dx} (x e^x) = 1 \cdot e^x + x \cdot e^x = e^x + x e^x$$

$$\frac{d}{dx} (x e^x - e^x) = e^x + x e^x - e^x = x e^x$$

$$\int x e^x dx = (x-1)e^x + C$$



Compare: triangle has area  $\frac{e}{2} \approx 1.36$

So ↓ does seem reasonable.

(May 4)

# Examples of the first form of the Fundamental Theorem of Calculus.

$$\frac{d}{dx} \int_a^x f(t) dt = f(x)$$

$$\frac{d}{dx} \int_0^1 (7t^2+1)e^t dt = 0$$

$$\frac{d}{dx} \int_3^x \underbrace{6t^2 e^{t^3-t}}_{f(t)} dt = 6x^2 e^{x^2-x}$$

$$\frac{d}{dx} e^7 = 0$$

$$\frac{d}{dx} e^7 = e^7 \cdot \frac{d7}{dx} = 0$$

$$\frac{d}{dx} \int_3^{x^3} \sin t^2 dt = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (\sin u^2) \cdot 3x^2 = 3x^2 \sin x^6$$

$$(\sin t)^2 = \sin^2 t$$

$$y = \int_3^u \sin t^2 dt, \quad u = x^3$$

$$\sin t^2 = \sin(t^2)$$

$$\frac{d}{dx} \int_x^2 e^{\sin t} dt = -\frac{d}{dx} \int_2^x e^{\sin t} dt = -e^{\sin x}$$

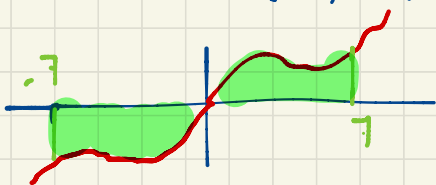
$$\int_0^{\cos x} f(t) dt = \int_0^{x^3} f(t) dt + \int_{x^3}^{\cos x} f(t) dt$$

$$\frac{d}{dx} \int_{x^3}^{\cos x} (t^2+3)^5 dt = \frac{d}{dx} \left[ \int_0^{\cos x} (t^2+1)^5 dt - \int_0^{x^3} (t^2+1)^5 dt \right] = (\cos^2 x + 1)^5 \cdot (-\sin x) - (x^3+1)^5 \cdot 3x^2$$

$$\int_{-1}^1 t(35t^4 + \cos t)^7 dt = 0$$

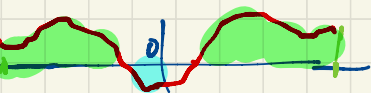
$f(t) = \underbrace{t}_{\text{odd}} \underbrace{(35t^4 + \cos t)^7}_{\text{even}}$  is an odd function

$$f(-t) = -f(t)$$



For an even function  $f(-x) = f(x)$ ,

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$$



For an odd function  $f(-x) = -f(x)$ ,

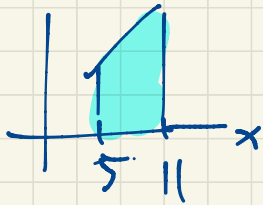
$$\int_{-a}^a f(x) dx = 0.$$

$$\int_{-1}^1 (\cancel{x^5} - 2x^4 - \cancel{x^3} + 7x^2 + 3\cancel{x} + 1) dx = 2 \int_0^1 (-2x^4 + 7x^2 + 1) dx$$

$$= 2 \left[ -\frac{2}{5}x^5 + \frac{7}{3}x^3 + x \right]_0^1 = 2 \left( -\frac{2}{5} + \frac{7}{3} + 1 \right) = 2 \left( \frac{-6 + 35 + 15}{15} \right)$$

$$= 2 \cdot \frac{44}{15} = \frac{88}{15}$$

$$\int_5^{11} (3x+1) dx = \int_5^{11} (3t+1) dt = \int_5^{11} (3u+1) du$$



$$\frac{d}{dx} \int_a^x f(t) dt = f(x)$$

$$\frac{d}{dx} \int_a^x f(s) ds = f(x)$$