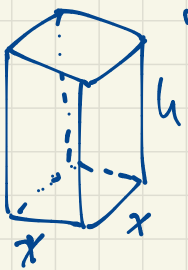
Math 2200-01 (Calculus I) Spring 2020

Book 3

Sec 4.5: Optimization

p.285 #19. Of all boxes with a square base and a volume $8\,m^3$, which one has the minimum surface area?



volume $V = x^2 h = 8 \implies h = \dfrac{8}{x^2}$

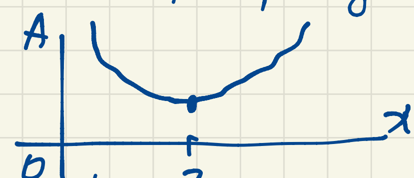
area $A = \underbrace{2x^2}_{\text{top and bottom}} + \underbrace{4xh}_{\text{sides}} = 2x^2 + 4x \cdot \dfrac{8}{x^2} = 2x^2 + \dfrac{32}{x}$, $x > 0$

The domain is $(0, \infty)$, an unbounded open interval.

$$\frac{dA}{dx} = 4x - \frac{32}{x^2} = \frac{4}{x^2}\left(x^3 - 8\right)$$

The critical point is at $x = 2$.
(Recall: critical points are where the derivative is zero or undefined. There are no points of the domain where $\frac{dA}{dx}$ is undefined, only one point where $\frac{dA}{dx} = 0$.)

For $0 < x < 2$, $\frac{dA}{dx} < 0$ so $A(x)$ is decreasing.
For $x = 2$, $\frac{dA}{dx} = 0$.
For $x > 2$, $\frac{dA}{dx} > 0$ so $A(x)$ is increasing.



So the minimum surface area $A(2) = 12\,m^2$ occurs for a box of size $2m \times 2m \times 2m$.

Sec 4.4. p.278 #21.

$$f(x) = (x-6)(x+6)^2 = (x^2-36)(x+6) = x^3 + 6x^2 - 36x - 216$$

$$f'(x) = 3x^2 + 12x - 36 = 3(x^2 + 4x - 12) = 3(x+6)(x-2)$$

$$f''(x) = 6x + 12 = 6(x+2)$$

$-6 \quad -2 \quad 2 \quad 6$

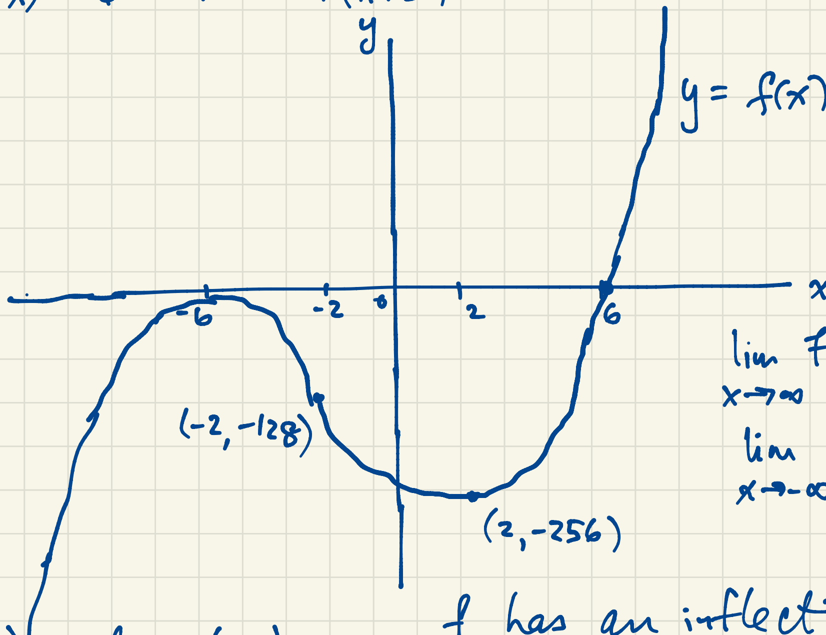
$f \quad 0 \qquad\qquad 0$

$f' \; + + 0 - - - - 0 + + + +$

$f'' \; - - - - 0 + + + + + + +$

$f(-2) = (-8)(4)^2 = -128$

$f(2) = (-4)(8^2) = -256$



$\lim_{x\to\infty} f(x) = \infty$

$\lim_{x\to-\infty} f(x) = -\infty$

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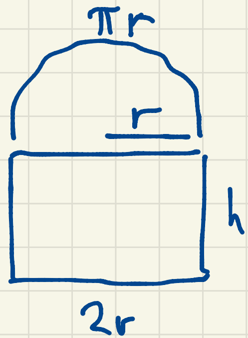
f is increasing on $(-\infty, -6)$ and on $(2, \infty)$,
decreasing on $(-6, 2)$,
concave down on $(-\infty, -2)$,
concave up on $(-2, \infty)$.

f has an inflection point $(-2, -128)$,
a local minimum point $(2, -256)$,
a local maximum point $(-6, 0)$,
no absolute extrema or asymptotes.

Sec 4.5 p. 287 #41. Let r be the radius of the semicircular window pane.



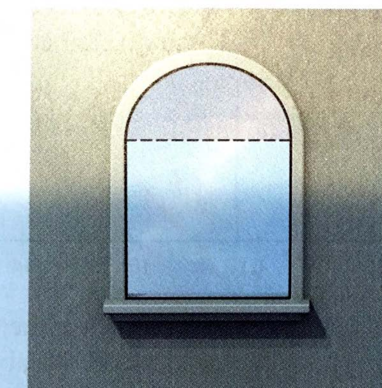
The perimeter is $P = \pi r + 2r + 2h = 20$

$$(\pi+2)r + 2h = 20$$

$$2h = 20 - \pi r - 2r$$

$$h = 10 - \frac{\pi+2}{2} r$$

41. A window consists of rectangular pane of glass surmounted by a semicircular pane of glass (see figure). If the perimeter of the window is 20 feet, determine the dimensions of the window that maximize the area of the window.



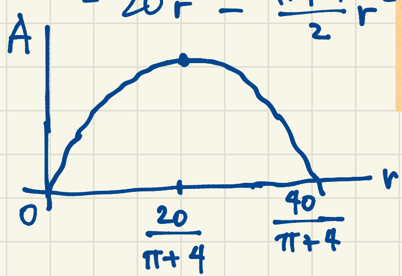
$$\frac{dA}{dr} = 20 - (\pi+4)r$$

$$A = \frac{\pi}{2} r^2 + 2rh$$

$$= \frac{\pi}{2} r^2 + 2r\left(10 - \frac{\pi+2}{2} r\right)$$

$$= 20r + \left(\frac{\pi}{2} - (\pi+2)\right) r^2$$

$$= 20r - \frac{\pi+4}{2} r^2$$



The critical point is at $r = \frac{20}{\pi+4}$.

When $0 < r < \frac{20}{\pi+4}$, $\frac{dA}{dr} > 0$ so A is increasing.

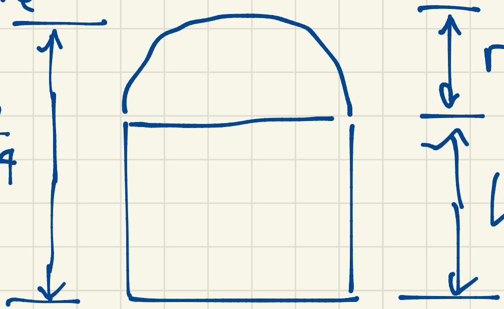
When $\frac{20}{\pi+4} < r < \frac{40}{\pi+4}$, $\frac{dA}{dr} < 0$ so A is decreasing.

So the maximum area occurs when $r = \frac{20}{\pi+4}$. Alternatively since $A \geq 0$ requires r to be in $\left[0, \frac{40}{\pi+4}\right]$, we need only check A at endpoints and the critical point.

$$A = \left(20 - \frac{\pi+4}{2} r\right) r$$

The dimensions of the window that maximizes the area are

$\frac{40}{\pi+4}$



$$r = \frac{20}{\pi+4}$$

$$h = 10 - \frac{\pi+2}{2} r = 10 - \frac{\pi+2}{2} \cdot \frac{20}{\pi+4}$$

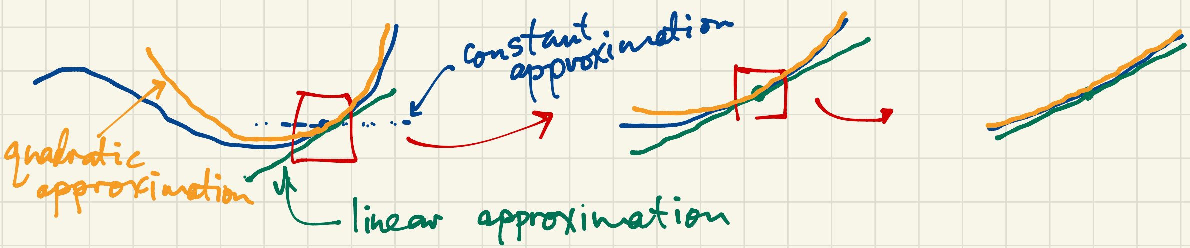
$$= 10 - \frac{10(\pi+2)}{\pi+4}$$

$$= \frac{10(\pi+4) - 10(\pi+2)}{\pi+4}$$

$$= \frac{20}{\pi+4}$$

$$2r = \frac{40}{\pi+4}$$

Sec 4.6. Linearization and Differentials



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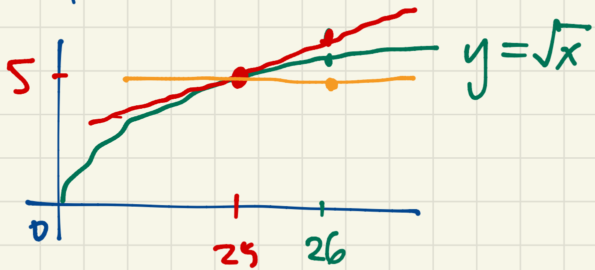
For x close to a, $f(x) \approx \underbrace{L(x)}_{\text{linearization of } f \text{ at } (a, f(a))} = f'(a)(x-a) + f(a) = \boxed{}\, x + \boxed{}$

approximately equal to

$\boxed{} = f'(a)$, $\quad \boxed{} = f(a) - af'(a)$

Example: Use the linearization of $\sqrt{x}$ at $(25, 5)$ to approximate $\sqrt{26}$.



$f(x) = \sqrt{x}$, $\quad f(25) = 5$

$f'(x) = \frac{1}{2\sqrt{x}}$, $\quad f'(25) = \frac{1}{2\sqrt{25}} = \frac{1}{10}$

The linearization of $\sqrt{x}$ at $(25, 5)$ is

$$L(x) = f'(25)(x-25) + f(25)$$
$$= \frac{1}{10}(x-25) + 5.$$

If $x \approx 25$ then $\sqrt{x} \approx \frac{1}{10}(x-25) + 5$.

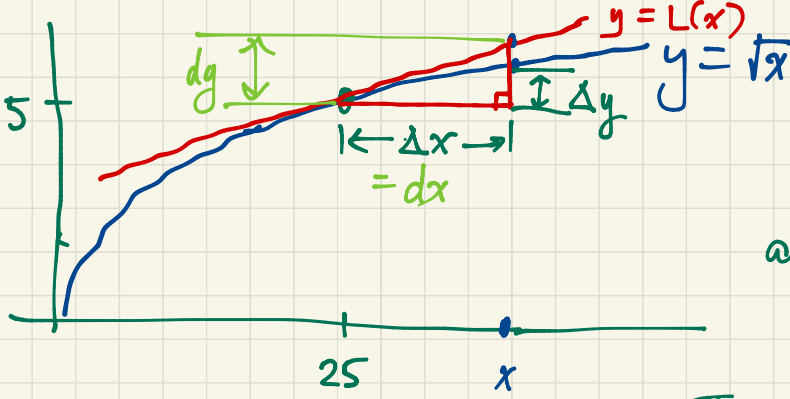
Eg. $\sqrt{26} \approx \frac{1}{10}(26-25) + 5 = 5.1$

Check: $\sqrt{26} \approx 5.099019514$

correct to 3 significant digits.

Cruder approximation: $\sqrt{26} \approx 5$. (correct to one significant digit)

Constant approximation $\sqrt{x} \approx 5$



If we move from $(25, 5)$ to $(x, f(x))$ on the graph, our actual function $f(x) = \sqrt{x}$ changes by an exact amount

$$\Delta y = f(x) - f(25).$$

The secant line from $(25, 5)$ to $(x, f(x))$ has slope $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x) - f(25)}{x - 25}$.

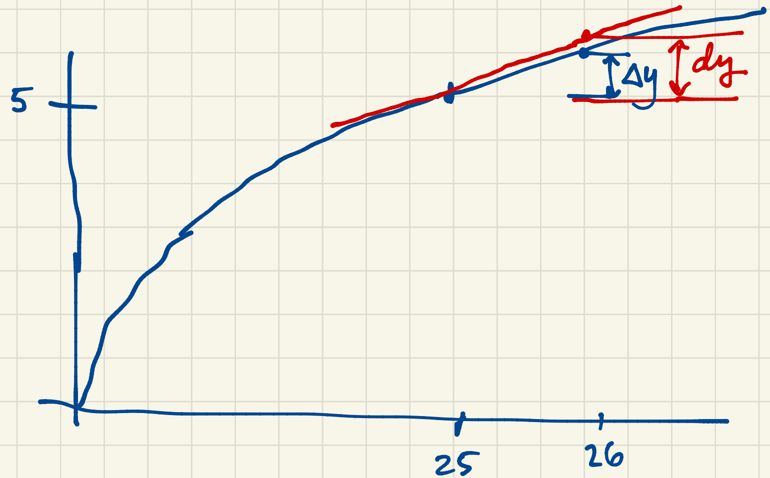
On the tangent line,

$$\frac{\Delta L(x)}{\Delta x} = \frac{L(x) - L(25)}{x - 25} = f'(25) = \frac{dy}{dx}$$

Until now we have written $\frac{dy}{dx}$ as an indivisible symbol. Now we are interpreting dx and dy as changes in x and y. They are changes on the tangent line (just like Δx and Δy are corresponding changes on the actual curve of f).

dx and dy are differentials.

We'll interpret $\sqrt{26} \approx 5.1$ in this new language:



$f'(25) = \frac{1}{10} = \frac{dy}{dx} \Rightarrow dy = \frac{1}{10} dx = \frac{1}{10} \times 1 = 0.1$

$dx = \Delta x = 26 - 25 = 1$

As we move from $x = 25$ to $x = 26$,
the corresponding change in y is

$\Delta y \approx dy = 0.1$.

So $\sqrt{26} \approx 5.1$

This is a quick and easy interpretation for differentials. We will be using differentials throughout Calculus.

Integrals $\int_a^b f(x)\, \underbrace{dx}$

$x \longrightarrow u \longrightarrow y$

$$\frac{dy}{dx} = \frac{dy}{\cancel{du}} \cdot \frac{\cancel{du}}{dx}$$

If $y = \sin x$, find $\frac{dy}{dx}$ and dy.

$\frac{dy}{dx} = \cos x$ so $dy = (\cos x)\, dx$

so $\Delta y \approx dy = \cos x \, dx$

Sec 4.7 l'Hôpital's Rule

The limit $\displaystyle\lim_{x\to\infty} \underbrace{\frac{x^3+2x^2-x-1}{2x^3-5x}}_{\text{"indeterminate form } \frac{\infty}{\infty}\text{"}} = \lim_{x\to\infty} \underbrace{\frac{1+\frac{2}{x}-\frac{1}{x^2}-\frac{1}{x^3}}{2-\frac{5}{x^2}}}_{\text{determinate form}} = \frac{1}{2}.$

The limit $\displaystyle\lim_{x\to\infty} \frac{1}{3x^2+5} = 0.$

"determinate form $\frac{1}{\infty}$"

Do not confuse l'Hôpital's Rule with the Quotient Rule!

$$\lim_{x\to 0} \underbrace{\frac{\sin x}{x}}_{\text{indeterminate form } \frac{0}{0}} = 1$$

<u>l'Hôpital's Rule</u> For limits of indeterminate form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, $\displaystyle\lim_{x} \frac{f(x)}{g(x)} = \lim_{x} \frac{f'(x)}{g'(x)}$ assuming the second limit exists.

eg. $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1.$

eg. $\displaystyle\lim_{x\to\infty} \frac{x^3+2x^2-x-1}{2x^3-5x} = \lim_{x\to\infty} \frac{3x^2+4x-1}{6x^2-5} = \lim_{x\to\infty} \frac{6x+4}{12x} = \lim_{x\to\infty} \frac{6}{12} = \frac{1}{2}$

eg. $\lim_{x\to\infty} \underline{xe^{-x}} = \lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0$

indeterminate form $\infty\cdot 0$ indeterminate form $\frac{\infty}{\infty}$

Don't keep using l'Hôpital's Rule beyond this point; that approach never reaches a conclusion.

eg. $\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} (-x) = 0.$

indeterminate form $0\cdot(-\infty)$

indeterminate $\frac{-\infty}{\infty}$

$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{x}{\frac{1}{\ln x}} = \lim_{x\to 0^+} \frac{1}{-\frac{1}{(\ln x)^2}\cdot\frac{1}{x}} = \lim_{x\to 0^+} \left(-x(\ln x)^2\right)$ this is not an improvement!

indeterminate form $\frac{0}{0}$

$(a-b)(a+b) = a^2-b^2$

$\lim_{x\to\infty} \left(\sqrt{x^2+10x} - x\right)\cdot\frac{\sqrt{x^2+10x}+x}{\sqrt{x^2+10x}+x} = \lim_{x\to\infty} \frac{x^2+10x-x^2}{\sqrt{x^2+10x}+x} = \lim_{x\to\infty} \frac{10x}{\sqrt{x^2+10x}+x}$

indeterminate form $\infty-\infty$

$= \lim_{x\to\infty} \frac{10}{\sqrt{1+\frac{10}{x}}+1} = \frac{10}{\sqrt{1+0}+1} = 5.$

Note:

$\sqrt{x^2+10x} = \sqrt{x^2\left(1+\frac{10}{x}\right)}$

$= x\sqrt{1+\frac{10}{x}}$

$\sqrt{x^2} = |x| = \begin{cases} x & \text{if } x\geq 0 \\ -x & \text{if } x<0 \end{cases}$

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Another l'Hôpital's Rule example for the indeterminate form 1^∞.

$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x$ where a is constant.

In order to use l'Hôpital's Rule, we need to rewrite this as a quotient.

$$b^c = (e^{\ln b})^c = e^{c\ln b}$$

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x = \lim_{x\to\infty} e^{x\ln\left(1+\frac{a}{x}\right)} = \lim_{u\to 1} e^{\frac{a}{u-1}\ln u} = e^{a\cdot 1} = e^a.$$

$u = 1+\frac{a}{x} \to 1.$

$u-1 = \frac{a}{x}$

$x = \frac{a}{u-1}$

$$\lim_{u\to 1}\frac{\ln u}{u-1} = \lim_{u\to 1}\frac{1/u}{1} = 1.$$

(indeterminate form $\frac{0}{0}$)

If $f(x) = e^{ax}$ then $\lim_{u\to 1} f\left(\frac{\ln u}{u-1}\right) = f\left(\lim_{u\to 1}\frac{\ln u}{u-1}\right) = f(1) = e^{a\cdot 1} = e^a.$

Does $\lim_{u} f(g(u)) = f\left(\lim_{u} g(u)\right)$? Can you move the limit inside?

We can do this when f is continuous. and g is continuous.

$c = \lim_{u} g(u)$, write $t = g(u) \to c$. $\lim_{t\to c} f(t) = f(c)$.

Some books define $e^a = \lim_{x\to\infty} \left(1+\frac{a}{x}\right)^x$.

This limit comes from compound interest.

If you deposit a principal amount A in the bank at nominal interest rate r (eg. 5% interest per annum would give $r=0.05$). If interest is compounded annually then after one year you have earned rA interest. The ~~total~~ balance in the bank after a year would be
$A + rA = (1+r)A$.

If interest is compounded semiannually (every 6 months) then after 6 months you have $\left(1+\frac{r}{2}\right)A$ as balance after 6 months; then at the end of the year you have

$$\left(1+\frac{r}{2}\right)\cdot\left(1+\frac{r}{2}\right)A = \left(1+\frac{r}{2}\right)^2 A = \Big(1+\underbrace{r+\frac{r^2}{4}}_{\text{effective rate of interest.}}\Big)A.$$

If interest is compounded n times per year then every $\frac{1}{n}$ year your balance is multiplied by $1+\frac{r}{n}$. After one year your balance is $\left(1+\frac{r}{n}\right)^n A$.

For continually compounded interest we let $n\to\infty$.

$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n A = e^r A$. Eg. 5% interest compounded continuously results in balance $e^{0.05}A \approx 1.05127\,A$ (effectively 5.127 % interest rate per annum).

p. 311 # 69. $\displaystyle\lim_{x\to\infty} \frac{\log_2 x}{\log_3 x} = \lim_{x\to\infty} \frac{\cancel{(\ln x)}/\ln 2}{\cancel{(\ln x)}/\ln 3} = \frac{\ln 3}{\ln 2} = \log_2 3$

$\log_2 x = \dfrac{\ln x}{\ln 2}$ why?

$a = \log_2 x \iff 2^a = x \iff \ln(2^a) = \ln x$

$\iff a \ln 2 = \ln x$

$\iff a = \dfrac{\ln x}{\ln 2}$.

$$\begin{tikzcd} x \arrow[r, bend left, "\log_2"] & a \arrow[l, bend left, "\exp_2"] \end{tikzcd}$$

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Sec 4.6 Linearization and Differentials

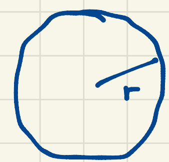
$y = f(x)$

$\dfrac{dy}{dx} = f'(x)$

$\Delta y \approx dy = f'(x)\,dx = f'(x)\,\Delta x$

small change in y due to change in x.

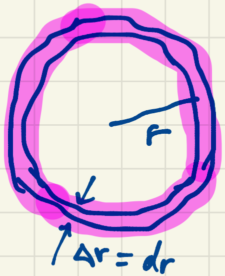
Formulas for area and circumference of a circle



$A = \pi r^2$

$C = 2\pi r$

From the formula $A = \pi r^2$ you can deduce the formula $C = 2\pi r$ using linearization.



Starting with a circle of radius r, area $A = \pi r^2$, enlarge the circle by expanding the radius by a small amount Δr.

$$\Delta A \approx C \, \Delta r \approx C \, \Delta r$$

$$\Delta A \approx dA = 2\pi r \, dr = 2\pi r \, \Delta r$$

$$A = \pi r^2$$

$$\frac{dA}{dr} = 2\pi r$$

$$dA = 2\pi r \, dr$$

So $C = 2\pi r$.

For a sphere of radius r,

$V = \frac{4}{3}\pi r^3$. What is the surface area?

Imagine painting the surface of a ball of radius r. How much paint is required to paint the surface? If I add a layer of paint of thickness Δr then the new volume changes by

$$\Delta V \approx A \, \Delta r$$

$$\Delta V \approx dV = 4\pi r^2 \, dr = 4\pi r^2 \, \Delta r$$

So $A = 4\pi r^2$.

Sec 4.6 p. 299 #28. $f(x) = \frac{x}{x+1}$. Approximate $f(1.1)$ using the linearization at $x=1$.

$f(1) = \frac{1}{2} = 0.5$. This value is the base for our linearization.

$$\frac{dy}{dx} = f'(x) = \frac{(x+1)\cdot 1 - x\cdot 1}{(x+1)^2} = \frac{1}{(x+1)^2}$$

$$dy = \frac{dx}{(x+1)^2}$$

$L(x) = f(a) + f'(a)(x-a)$, $a=1$

$= \frac{1}{2} + \frac{1}{4}(x-1)$

$f(1.1) \approx L(1.1) = \frac{1}{2} + \frac{1}{4}(0.1) = 0.5 + 0.025 = 0.525$

Alternatively using differentials:

x goes from 1 to 1.1, a small change of $\Delta x = dx = 0.1$. What is the corresponding change in y?

$$\Delta y \approx dy = \frac{dx}{(x+1)^2}$$

$$\Delta y \approx \frac{\Delta x}{(1+1)^2} = \frac{\Delta x}{4} = \frac{0.1}{4} = 0.025$$

$$y \approx 0.5 + \Delta y = 0.525.$$

Actually $0.0023 = 0.23\%$

"per cent" means "divided by 100"

$f(1.1) = \frac{1.1}{2.1} = 0.52380952\ldots$

The error is $0.525 - f(1.1) = 0.00119\ldots$ (an over estimate since this difference is positive).

The relative error is $\frac{0.00119}{0.5238} = 0.0023\ldots$ The percentage error is $0.0023 \times 100\% = 0.23\%$

(about a quarter of a percent error).

Sec 4.9 Antiderivatives

$$\frac{d}{dx}(7x^2+2x+1)^3 = 3(7x^2+2x+1)^2(14x+2).$$

Reverse: ~~The~~ An antiderivative of $3(7x^2+2x+1)^2(14x+2)$ is $(7x^2+2x+1)^3$.

The general answer for antiderivative: $(7x^2+2x+1)^3$ + constant.

$$\frac{d}{dx}\left[(7x^2+2x+1)^3+11\right] = 3(7x^2+2x+1)^2(14x+2).$$

We write $\displaystyle\int 3(7x^2+2x+1)^2(14x+2)\,dx = (7x^2+2x+1)^3+C$ where C is an arbitrary constant.

The general antiderivative of $3(7x^2+2x+1)^2(14x+2)$ with respect to x is $(7x^2+2x+1)^3$ + constant.

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eg. $\displaystyle\int x^k\,dx = \begin{cases} \frac{1}{k+1}x^{k+1}, & \text{if } k \neq -1 \\ \ln|x|, & \text{if } k=-1 \end{cases}$

Check: $\displaystyle\frac{d}{dx}\frac{x^{k+1}}{k+1} = \frac{\cancel{(k+1)}x^k}{\cancel{k+1}} = x^k$ (k constant).

$$\frac{d}{dx}(\ln|x|) = x^{-1} = \frac{1}{x}$$

$$\int (x^3-7x^2-5x+3)\,dx = \frac{1}{4}x^4-\frac{7}{3}x^3-\frac{5}{2}x^2+3x+C$$

Check: $\frac{d}{dx}\left(\frac{1}{4}x^4-\frac{7}{3}x^3-\frac{5}{2}x^2+3x+C\right) = x^3-7x^2-5x+3$

Compare notation: $\int f(x)\,dx$ antiderivative (function) (indefinite integral)

vs. $\int_a^b \underbrace{f(x)}_{\text{integrand}}\,dx$ integral (a number) (definite integral)

How do we find antiderivatives? Best method: guess and check.

Eg. $\int x\sin x\,dx = \sin x - x\cos x + C$

$\frac{d}{dx}(?) = x\sin x$

$\frac{d}{dx}(-\cos x) = \sin x$

$\frac{d}{dx}(x\cos x) = 1\cdot\cos x + x\cdot(-\sin x) = \cos x - x\sin x$

$\frac{d}{dx}(x\cos x - \sin x) = (\cancel{\cos x} - x\sin x) - \cancel{\cos x}$

$= -x\sin x$

$\frac{d}{dx}(\sin x - x\cos x) = x\sin x$

Sec 4.9 #92. Given velocity $v(t) = e^t + 4$ and initial position $s(0) = 2$ $\Big\}$ initial value problem

find $s(t)$, the position at time t.

Recall $s'(t) = v(t)$ so $s(t) = \int v(t)\,dt = \int (e^t + 4)\,dt = e^t + 4t + C$

$$\begin{tikzcd} s(t) \arrow[r, bend left, "\text{derivative}"] & v(t) \arrow[l, bend left, "\text{antiderivative}"] \end{tikzcd}$$

Using $s(0) = 2$, find C.

$s(t) = e^t + 4t + C$

$2 = s(0) = 1 + 0 + C \Rightarrow C = 1$

So $s(t) = e^t + 4t + 1$.

#97. $a(t) = -32$ $\}$ differential equation

$v(0) = 20$, $s(0) = 0$ $\}$ initial conditions

$\Big\}$ initial value problem

$$\begin{tikzcd} s(t) \arrow[r, bend left, "\text{derivative}"] & v(t) \arrow[l, bend left, "\text{antiderivative}"] \arrow[r, bend left, "\text{derivative}"] & a(t) \arrow[l, bend left, "\text{antiderivative}"] \end{tikzcd}$$

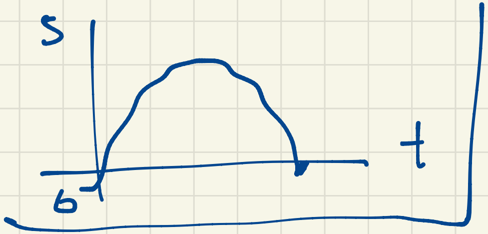
$v(t) = \int a(t)\,dt = \int (-32)\,dt = -32t + C$

$20 = v(0) = C \Rightarrow v(t) = -32t + 20$

$s(t) = \int v(t)\,dt = \int (-32t + 20)\,dt$

$= -16t^2 + 20t + K$. Since $s(0) = 0 = K$,

$s(t) = -16t^2 + 20t$.



#100. $a(t) = 2\cos t$

$v(0) = 1$

$s(0) = 0$

$v(t) = \int a(t)\,dt = \int 2\cos t\,dt = 2\sin t + C$

$1 = v(0) = C \Rightarrow v(t) = 2\sin t + 1$

$s(t) = \int v(t)\,dt = \int (2\sin t + 1)\,dt$

$= -2\cos t + t + K$

$0 = s(0) = -2 + K \Rightarrow K = 2$

$s(t) = -2\cos t + t + 2$

$\int \frac{dt}{t^2+1} = \int \frac{1}{t^2+1}\,dt = \tan^{-1} t + C$

$\cancel{\int \tan^{-1} t\,dt = \frac{1}{t^2+1}}$

$\frac{d}{dt}\frac{1}{t^2+1} = -\frac{1}{(t^2+1)^2}\cdot 2t = -\frac{2t}{(t^2+1)^2}$

$\int \frac{dt}{t} = \int \frac{1}{t}\,dt = \ln|t| + C$

$\int dt = \int 1\,dt = t + C$

April 24

Brushing up on l'Hôpital's Rule

$$\lim_{x\to\infty} \frac{x^n}{e^x} = 0 \qquad \text{(n applications of l'Hôpital's Rule)}$$

Eg for $n=4$, $\lim_{x\to\infty} \frac{x^4}{e^x} = \lim_{x\to\infty} \frac{4x^3}{e^x} = \lim_{x\to\infty} \frac{12x^2}{e^x} = \lim_{x\to\infty} \frac{24x}{e^x} = \lim_{x\to\infty} \frac{24}{e^x} = 0.$

$$\lim_{x\to\infty} \frac{x^4}{e^{x^2}} = \lim_{x\to\infty} \frac{4x^3}{2xe^{x^2}} = \lim_{x\to\infty} \frac{2x^2}{e^{x^2}} = \cdots = 0.$$

$e^{x^2} = e^{(x^2)}$ or ~~$(e^x)^2$~~

$(e^x)^2 = e^{2x}$

$$\lim_{x\to\infty} \frac{x^4}{e^{x^2}} = \lim_{u\to\infty} \frac{u^2}{e^u} = 0$$

$u = x^2$

OR: $0 < \frac{x^4}{e^{x^2}} < \frac{x^4}{e^x}$, $\quad 0 \to 0$, $\frac{x^4}{e^x} \to 0$

$\lim_{x\to\infty} \frac{x^4}{e^{x^2}} = 0$ by the Squeeze Theorem.

p.311 #28. $\lim_{x\to0^+} \frac{x-3\sqrt{x}}{x-\sqrt{x}} = \lim_{x\to0^+} \frac{1-\frac{3}{2\sqrt{x}}}{1-\frac{1}{2\sqrt{x}}} = \lim_{x\to0^+} \frac{2\sqrt{x}-3}{2\sqrt{x}-1} = 3.$

OR $\lim_{x\to0^+} \frac{x-3\sqrt{x}}{x-\sqrt{x}} = \lim_{x\to0^+} \frac{\sqrt{x}-3}{\sqrt{x}-1} = \frac{0-3}{0-1} = 3.$

#40. $\lim_{x\to 0} \frac{\sin x - x}{7x^3} = \lim_{x\to 0} \frac{\cos x - 1}{21x^2} = \lim_{x\to 0} \frac{-\sin x}{42x} = \lim_{x\to 0} \frac{-\cos x}{42} = -\frac{1}{42}$

#83. $\lim_{x\to 0} (x+\cos x)^{1/x} = \lim_{x\to 0} e^{\frac{1}{x}\ln(x+\cos x)} = e^1 = e$.

indeterminate form 1^∞

using $e^{\ln a} = a$, $e^{\ln(x+\cos x)} = (x+\cos x)$

Compare: $\lim_{x\to 0^+} (x+1)^{1/x} = \lim_{t\to\infty} \left(1+\frac{1}{t}\right)^t = e$

$t = \frac{1}{x} \to \infty$

$\lim_{x\to 0} \frac{\ln(x+\cos x)}{x} = \lim_{x\to 0} \frac{\frac{1}{x+\cos x}(1-\sin x)}{1} = 1$

indeterminate form $\frac{0}{0}$

#20. $\lim_{x\to 0} \frac{e^x - 1}{2x+5} = \frac{0}{5} = 0$.

Also: $\lim_{x\to\infty} \frac{\sin x}{x} = 0$.

$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$

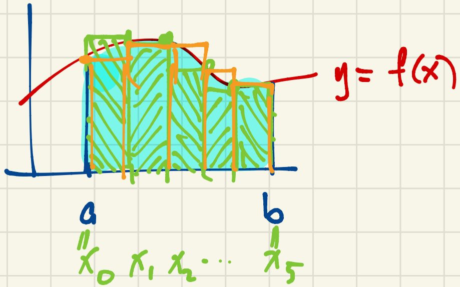
0

#68. $\lim_{x\to 0^+} x^{\frac{1}{\ln x}} = \lim_{x\to 0^+} e^{\frac{1}{\ln x}\ln x} = \lim_{x\to 0^+} e = e$.

$x = e^{\ln x} \Rightarrow x^{\frac{1}{\ln x}} = \left(e^{\ln x}\right)^{\frac{1}{\ln x}} = e$

Sec 5.1

(April 27)



Find the area shaded (under the graph of f, above the x-axis, for $a \le x \le b$).

Idea: Take large n. Divide $[a,b]$ into n subintervals each width $\Delta x = \frac{b-a}{n}$. Mark off points $x_0 = a$, x_1, x_2, ..., $x_n = b$ equally spaced Δx apart. i.e. $x_i = a + i\Delta x$.

The Right Riemann Sum with n=5 approximates the shaded area as

$$f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x = \sum_{i=1}^{5} f(x_i)\Delta x$$

(the sum of $f(x_i)\Delta x$ as i ranges from 1 to 5)

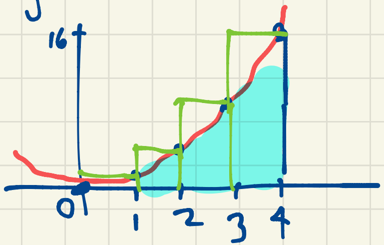
The left Riemann Sum with n=5 subintervals is

$$\sum_{i=1}^{5} f(x_{i-1})\Delta x = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x.$$

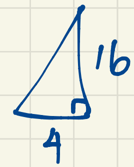
The midpoint Riemann Sum with n=5 subintervals is

$$\sum_{i=1}^{5} f\Big(\underbrace{\frac{x_{i-1}+x_i}{2}}_{\text{midpoint of the } i^{th} \text{ subinterval } [x_{i-1}, x_i]}\Big)\Delta x = f\Big(\frac{x_0+x_1}{2}\Big)\Delta x + f\Big(\frac{x_1+x_2}{2}\Big)\Delta x + \cdots + f\Big(\frac{x_4+x_5}{2}\Big)\Delta x$$

Eg. We want the area of $0 \le y \le x^2$, $0 \le x \le 4$.



Compare:



triangle has area $\frac{1}{2} \times 4 \times 16 = 32$.

So our area should be less than 32.

The Right Riemann Sum approximation with $n=4$ subintervals:

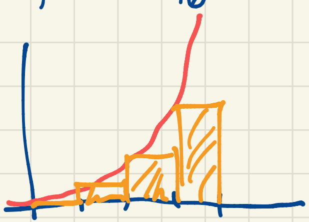
$$\sum_{i=1}^{4} f(x_i)\Delta x = 1\cdot 1 + 4\cdot 1 + 9\cdot 1 + 16\cdot 1 = 30.$$

x	$f(x)$
0	0
1	1
2	4
3	9
4	16

$$\Delta x = \frac{4-0}{4} = 1$$

The left Riemann Sum approximation with $n=4$ subintervals is

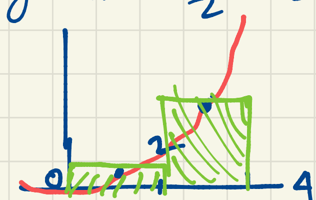
$$\sum_{i=1}^{4} f(x_{i-1})\Delta x = 0\cdot 1 + 1\cdot 1 + 4\cdot 1 + 9\cdot 1 = 14.$$



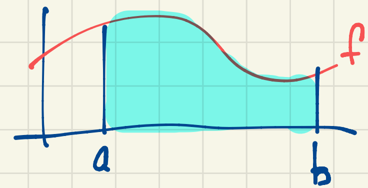
the true area is between 14 and 30. (The average is $\frac{30+14}{2} = 22$.)

The midpoint Riemann Sum with $n=2$ subintervals, $\Delta x = \frac{4-0}{2} = 2$:

$$\sum_{i=1}^{2} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x = f\left(\frac{0+2}{2}\right)2 + f\left(\frac{2+4}{2}\right)2 = f(1)\cdot 2 + f(3)\cdot 2 = 1\cdot 2 + 9\cdot 2 = 20.$$



A Riemann Sum approximation to the area with n subintervals of width $\Delta x = \frac{b-a}{n}$ has the form



$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

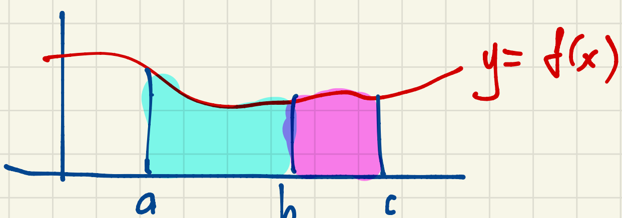
pick a point x_i^* in the i^{th} subinterval $[x_{i-1}, x_i]$

Now let $n \to \infty$.

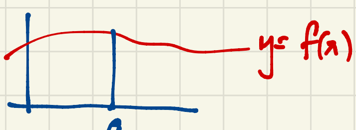
$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \int_a^b f(x)\,dx,$$ the Riemann integral of f from a to b.

April 28

Sec 5.2 (mostly) Properties of Integrals



$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



$$\int_a^a f(x)\,dx = 0$$

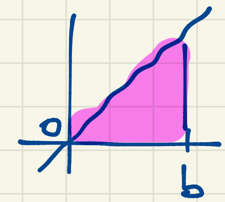
$$\int_a^b f(x)\,dx + \int_b^a f(x)\,dx = \int_a^a f(x)\,dx = 0$$

so $$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x \qquad \Delta x = \frac{b-a}{n}$$

$\uparrow$ or x_i^*

$$\boxed{\int_0^b x\,dx = \frac{b^2}{2}}$$

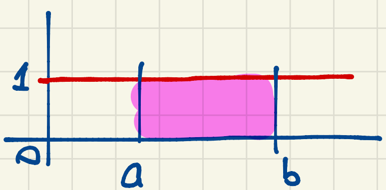


$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

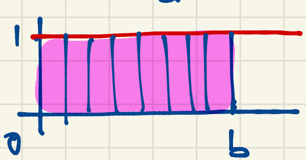
$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

c = constant

$$\int_a^b 1\,dx = \int_a^b dx = b-a$$



$$\underbrace{\int_0^a dx}_{a} + \underbrace{\int_a^b dx}_{b-a} = \underbrace{\int_0^b dx}_{b}$$



$$\int_0^b dx = b$$

$$\int_0^b 1\,dx = b$$

$$\int_0^b x\,dx = \frac{b^2}{2}$$

$$\int_0^b x^2\,dx = \frac{b^3}{3}$$

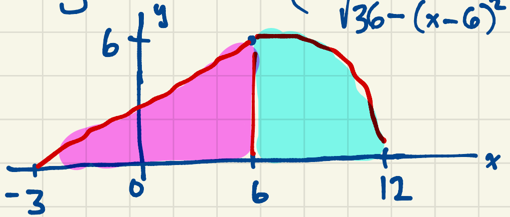
$$\int_0^b x^k\,dx = \frac{b^{k+1}}{k+1}$$

if $k \neq -1$

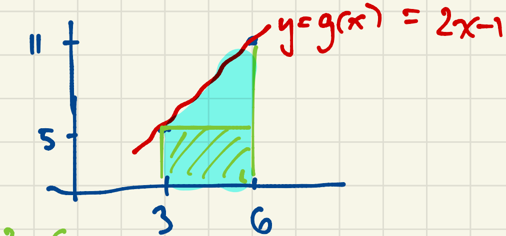
Compare:

$$\int x^k\,dx = \frac{x^{k+1}}{k+1} \text{ if } k\neq -1$$

Eg. $f(x) = \begin{cases} \frac{3}{2}(x+3), & \text{for } -3 \le x \le 6 \\ \sqrt{36-(x-6)^2} \end{cases}$



$$\int_{-3}^{12} f(x)\,dx = \frac{1}{2}\times 9\times 6 + \frac{36\pi}{4} = 27 + 9\pi$$



$$\int_3^6 g(x)\,dx$$

$$= 3\times 5 + \frac{1}{2}\times 3\times 6$$

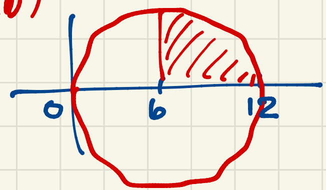
$$= 15 + 9 = 24.$$

circle

center at $(6,0)$

radius 6

$(x-6)^2 + y^2 = 36 = 6^2$



Second way: $\int_3^6 g(x)\,dx = \underbrace{3}_{\text{base}} \times \underbrace{\frac{5+11}{2}}_{\text{average height}} = 3\times 8 = 24.$

Third way: $y = g(x)$ has slope $\frac{11-5}{6-3} = \frac{6}{3} = 2$

Its equation is $y - 5 = 2(x-3)$

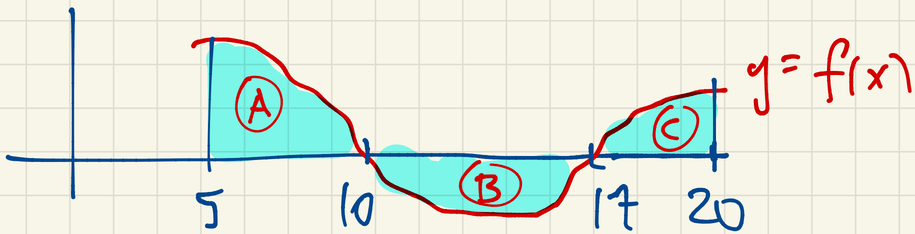
i.e. $y = 2x - 1$

$$\int_0^6 g(x)\,dx = \int_0^6 (2x-1)\,dx = 2\int_0^6 x\,dx - \int_0^6 dx$$

$$= 2\cdot\frac{6^2}{2} - 6 = 30$$

$$\int_3^6 g(x)\,dx = \int_0^6 g(x)\,dx - \int_0^3 g(x)\,dx \qquad \int_0^3 g(x)\,dx = 2\cdot\frac{3^2}{2} - 3 = 6$$

$$= 30 - 6 = 24.$$



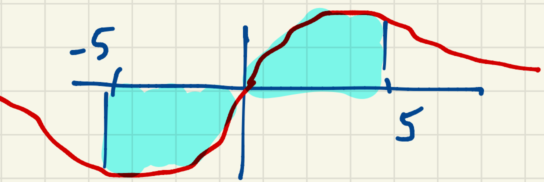
$$\int_5^{20} f(x)\,dx = \int_5^{10} f(x)\,dx + \int_{10}^{17} f(x)\,dx + \int_{17}^{20} f(x)\,dx$$

area of (A)

−(area of (B))

area of (C)

Eg. $\displaystyle\int_{-5}^{5} \frac{x}{x^{10}+1}\,dx = 0$ by symmetry



(April 29

Sec 5.3 Fundamental Theorem of Calculus

Let a, x be points in an interval where f is defined. Look at

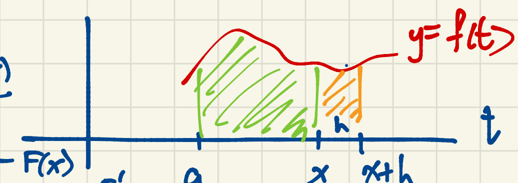
Let $F(x) = \int_a^x f(t)\,dt$. Then $F'(x) = f(x)$.

$F(x+h) \approx F(x) + h f(x)$

$f(x) \approx \dfrac{F(x+h) - F(x)}{h}$

Why? $\underbrace{\int_a^{x+h} f(t)\,dt}_{F(x+h)} = \underbrace{\int_a^x f(t)\,dt}_{F(x)} + \underbrace{\int_x^{x+h} f(t)\,dt}_{\approx\, h f(x)}$

$f(x) = \lim\limits_{h \to 0} \dfrac{F(x+h) - F(x)}{h} = F'(x)$



$$\begin{tikzcd} f \arrow[r, bend left, "\text{antiderivative}"] & F + C \arrow[l, bend left, "\text{derivative}"] \end{tikzcd}$$

$$F(x) = \int_a^x f(t)\,dt$$

$$F'(x) = f(x)$$

This is one form of the Fundamental Theorem of Calculus.

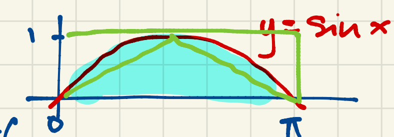
$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

In a nutshell: If you start with f and you integrate, then differentiate, you get back the original f.

The other way around gives the other form of the Fundamental Theorem of Calculus: If you start with a function F, take its derivative, then integrate, you get back $F + C$.

eg. Find the shaded area under the first arch of the sine curve:

$$\text{Area} = \int_0^\pi f(x)\,dx = F(\pi) = 1 - \cos\pi = 1-(-1) = 2 \qquad F(x) = 1 - \cos x$$



Use $F(x) = \int_0^x \sin t\,dt$, $\quad F'(x) = \sin x \Rightarrow F(x) = -\cos x + C$

$$0 = F(0) = -\cos 0 + C \Rightarrow C = 1$$

$\frac{\pi}{2} < \text{Area} < \pi$

The mechanics of using the Fundamental Theorem of Calculus to get an integral: Evaluate the antiderivative between the two endpoints.

$$\int_0^{\pi} \sin x \, dx = F(\pi) - F(0) = \underbrace{-\cos(\pi)}_{F(\pi)} - (\underbrace{-\cos 0}_{F(0)}) = 1 + 1 = 2.$$

First find an antiderivative $F(x)$ for $\sin x$. I can use $F(x) = -\cos x$.
(Any antiderivative will do because the $+C$ will cancel in the end.

In practice we write:

$$\int_0^{\pi} \sin x \, dx = \Big[-\cos x\Big]_0^{\pi} = -\cos\pi - (-\cos 0) = 1 + 1 = 2.$$

Notation:

$$\Big[F(x)\Big]_a^b = F(b) - F(a).$$

eg. $\Big[e^{x^2}\Big]_1^3 = e^9 - e$

Note: If $F(x) = \int f(x)\,dx$ (antiderivative)

then $\Big[F(x)\Big]_a^b = F(b) - F(a) = \int_a^b f(x)\,dx$

i.e. $\Big[\int f(x)\,dx\Big]_a^b = \int_a^b f(x)\,dx$

Eg. $\int_1^3 (6x^2+4x+3)\,dx = \left[2x^3+2x^2+3x\right]_1^3 = \left[2\times27+2\times9+9\right]-\left[2+2+3\right]$

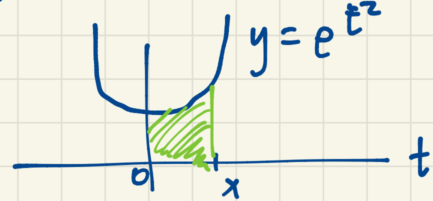
$= 81-7 = 74.$

$\int_1^3 e^{x^2}\,dx =$

Find an antiderivative for e^{x^2}? No nice formula for it.

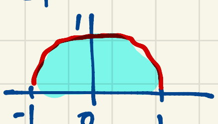
I know there is an antiderivative for e^{x^2}. Here it is:

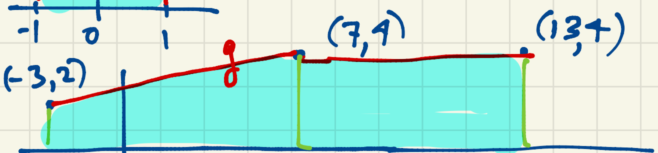
$F(x) = \int_0^x e^{t^2}\,dt$

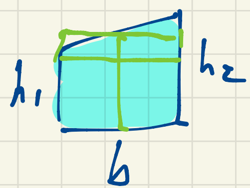


May 1

$\int_{-1}^1 \sqrt{1-x^2}\,dx = \frac{\pi}{2}$ using high school geometry.





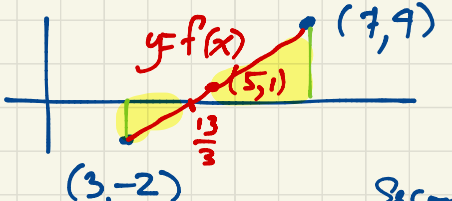


Area of trapezoid is $\frac{h_1+h_2}{2}b$

$\frac{4+2}{2}=3$

$\int_{-3}^{13} g(x)\,dx = 10\times3 + 6\times4$

$= 54$



midpoint Riemann sum with one subdivision

$$\int_3^7 f(x)\,dx = 4\times 1 = 4.$$

$$\text{slope} = \frac{6}{4} = \frac{3}{2}$$

$$y+2 = \frac{3}{2}(x-3)$$

$$y = \frac{3}{2}x - \frac{13}{2}$$

Second solution:

x-intercept is: $\frac{13}{3}$

solve $0 = \frac{3}{2}x - \frac{13}{2}$

$$\int_3^7 f(x)\,dx = \begin{pmatrix}\text{Area above}\\ x\text{-axis}\end{pmatrix} - \begin{pmatrix}\text{Area below}\\ x\text{-axis}\end{pmatrix}$$

$$= \frac{\left(7-\frac{13}{3}\right)\times 4}{2} - \frac{\left(\frac{13}{3}-3\right)\times 2}{2}$$

$$= \frac{\frac{8}{3}\times 4}{2} - \frac{\frac{4}{3}\times 2}{2} = \frac{16}{3} - \frac{4}{3} = \frac{12}{3} = 4.$$

Third solution: Use the Fundamental Theorem of Calculus.

$$\int_3^7 \left(\frac{3}{2}x - \frac{13}{2}\right)dx = \Big[\underbrace{\frac{3}{4}x^2 - \frac{13}{2}x}_{F(x)}\Big]_3^7 = \underbrace{\left[\frac{3\times 49}{4} - \frac{13\times 7}{2}\right]}_{F(7)} - \underbrace{\left[\frac{3\times 9}{4} - \frac{13\times 3}{2}\right]}_{F(3)}$$

$$= \left[\frac{147}{4} - \frac{91}{2}\right] - \left[\frac{27}{4} - \frac{39}{2}\right]$$

$$= \frac{120}{4} - \frac{52}{2} = 30 - 26$$

$$= 4.$$

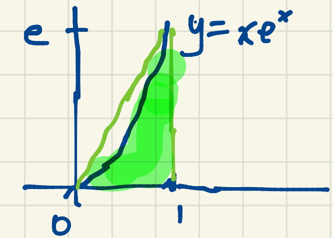
$$\int_1^3 (6x^2-5x+2)\,dx = \left[2x^3 - \frac{5}{2}x^2 + 2x\right]_1^3 = \left[54 - \frac{45}{2} + 6\right] - \left[2 - \frac{5}{2} + 2\right]$$

$$= 56 - \frac{45}{2} + \frac{5}{2} = 56 - \frac{40}{2} = 56-20$$

$$= 36.$$

$$\int x^k\,dx = \frac{x^{k+1}}{k+1} \quad \text{if } k \neq -1$$

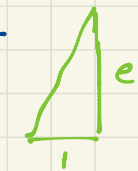
$$\int_0^1 xe^x\,dx = (x-1)e^x \Big|_0^1 = 0 - (-1) = 1.$$



Use the Fundamental Theorem of Calculus:

$$\frac{d}{dx}(?) = xe^x \quad ?$$

Compare: triangle



has area $\frac{e}{2} \approx 1.36$

So 1 does seem reasonable.

Guess and check:

$$\frac{d}{dx}(xe^x) = 1 \cdot e^x + x \cdot e^x = e^x + xe^x$$

(May 4

$$\frac{d}{dx}(xe^x - e^x) = e^x + xe^x - e^x = xe^x$$

$$\int xe^x\,dx = (x-1)e^x + C$$

Examples of the first form of the Fundamental Theorem of Calculus

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

$$\frac{d}{dx}\underbrace{\int_0^1 (7t^2+1)e^t\,dt}_{} = 0$$

$$\frac{d}{dx}\,e^7 = 0$$

$$\frac{d}{dx}\,e^7 = e^7\cdot\frac{d7}{dx} = 0$$

$$\frac{d}{dx}\int_3^x \underbrace{6t^2e^{t^3-t}}_{}\,dt = 6x^2e^{x^2-x}$$

$$f(t) = 6t^2e^{t^3-t}$$

$$\frac{d}{dx}\underbrace{\int_3^{x^3}\sin t^2\,dt}_{y} = \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = (\sin u^2)\cdot 3x^2 = 3x^2\sin x^6$$

u

$$(\sin t)^2 = \sin^2 t$$

$$y = \int_3^u \sin t^2\,dt, \quad u = x^3$$

$$\sin t^2 = \sin(t^2)$$

$$\frac{d}{dx}\int_x^2 e^{\sin t}\,dt = -\frac{d}{dx}\int_2^x e^{\sin t}\,dt = -e^{\sin x}$$

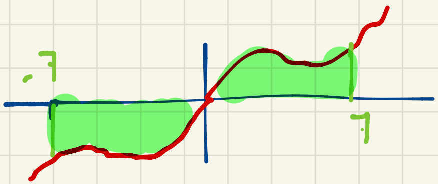
$$\int_0^{\cos x} f(t)\,dt = \int_0^{x^3} f(t)\,dt + \int_{x^3}^{\cos x} f(t)\,dt$$

$$\frac{d}{dx}\int_{x^3}^{\cos x}(t^2+3)^5\,dt = \frac{d}{dx}\left[\int_0^{\cos x}(t^2+1)^5\,dt - \int_0^{x^3}(t^2+1)^5\,dt\right] = (\cos^2 x+1)^5\cdot(-\sin x) - (x^6+1)^5\cdot 3x^2$$

$$\int_{-7}^{7} \underbrace{t(35t^4+\cos t)^7}\,dt = 0$$

$$f(t) = \underbrace{t}_{\text{odd}}(\underbrace{35t^4+\cos t}_{\text{even}})^7$$ is an odd function

$$f(-t) = -f(t)$$

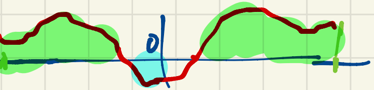


For an even function $f(-x) = f(x)$,

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

For an odd function $f(-x) = -f(x)$,

$$\int_{-a}^{a} f(x)\,dx = 0.$$

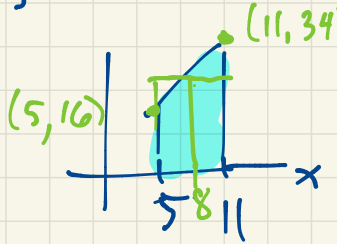


$$\int_{-1}^{1} (\cancel{x^5} - 2x^4 - \cancel{x^3} + 7x^2 + \cancel{3x} + 1)\,dx = 2\int_{0}^{1} (-2x^4 + 7x^2 + 1)\,dx$$

$$= 2\left[-\frac{2}{5}x^5 + \frac{7}{3}x^3 + x\right]_0^1 = 2\left(-\frac{2}{5} + \frac{7}{3} + 1\right) = 2\left(\frac{-6+35+15}{15}\right)$$

$$= 2\cdot\frac{44}{15} = \frac{88}{15}$$

$$\int_5^{11}(3x+1)\,dx = \int_5^{11}(3t+1)\,dt = \int_5^{11}(3u+1)\,du$$



Area $= \frac{1}{2}(16+34)\cdot 6 = \underbrace{25}\cdot\underbrace{6}$

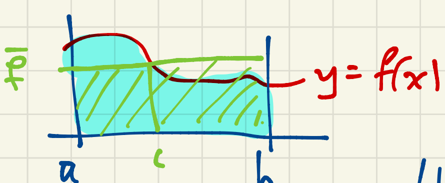
The average value of $3x+1$ on $[5,11]$ is 25.

For $f(x) = 3x+1$, $\bar{f} = 25$, $f(8) = 25$.

$$\boxed{\frac{d}{dx}\int_a^x f(t)\,dt} = f(x)$$

$$\frac{d}{dx}\int_a^x f(s)\,ds = f(x)$$

May 5



Sec 5.4

$\bar{f}$ = average value of f on $[a,b]$

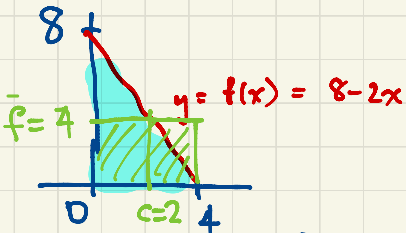
$$(b-a)\,\bar{f} = \int_a^b f(x)\,dx \iff \bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Mean Value Theorem for Integrals: If f is continuous on $[a,b]$, then there exists a point c in $[a,b]$ where $f(c) = \bar{f}$.

p. 386 #39. Find all points in $[0,4]$ where the function $f(x) = 8-2x$ has its average value.

First find $\bar{f} = \frac{1}{4-0}\int_0^4 (8-2x)\,dx = \frac{1}{4}\left[8x-x^2\right]_0^4 = \frac{1}{4}(16-0) = 4$

Find all points c in $[0,4]$ where $8-2c = f(c) = \bar{f} = 4 \Rightarrow 2c = 4 \Rightarrow c = 2$. (the midpoint of the interval)



Area $= \frac{1}{2} \times 4 \times 8 = 16$

$\bar{f} = \frac{16}{4} = 4$

p.385 #12. $\int_{-200}^{200} 2x^5\,dx = 0$.

If you used an antiderivative,

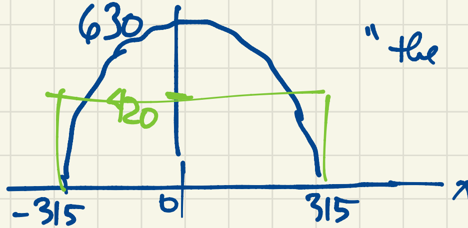
$$\int_{-200}^{200} 2x^5\,dx = \frac{x^6}{3}\Big|_{-200}^{200} = 0.$$

#23. $\int_{-2}^{2} \frac{x^3-4x}{x^2+1}\,dx = 0.$

#17. $\int_{-2}^{2} (\overbrace{\cancel{x^9} - \cancel{3x^5}}^{\text{odd function}} + \underbrace{2x^2-10}_{\text{even function}})\,dx = 2\int_0^2 (2x^2-10)\,dx = 2\left[\frac{2}{3}x^3 - 10x\right]_0^2 = 2\left(\frac{16}{3}-20\right) - 0$

$= 2\,\frac{16-60}{3} = -\frac{88}{3}.$

p.386 #47.



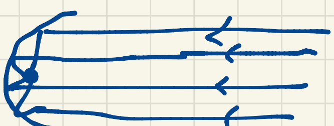
"the Gateway Arch" $y = 630\left(1-\left(\frac{x}{315}\right)^2\right)$

Average height: $\frac{1}{\cancel{630}}\int_{-315}^{315} \cancel{630}\left(1-\left(\frac{x}{315}\right)^2\right)dx = \int_{-315}^{315}\left(1-\left(\frac{x}{315}\right)^2\right)dx = 2\int_{0}^{315}\left(1-\left(\frac{x}{315}\right)^2\right)dx$

$\frac{x^2}{315^2}$

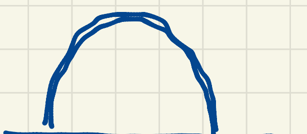
$= 2\left[x - \frac{x^3}{3\cdot 315^2}\right]_0^{315} = \frac{2}{3\cdot 315^2}\left[3\cdot 315^3 - 315^3\right] = \frac{2\cdot 2\cdot 315}{3} = 420$ ft.

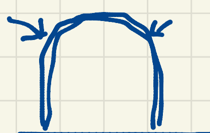
May 6

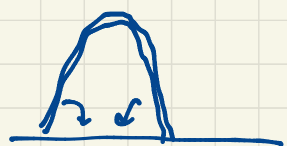


Catenary $y = \cosh x = \frac{e^x + e^{-x}}{2}$

is the shape of a hanging cable







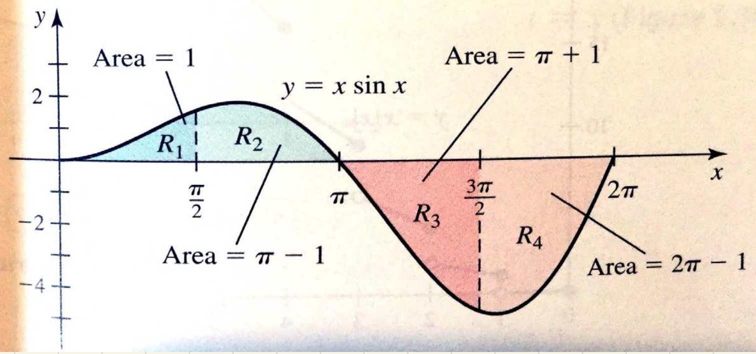
Sec 5.1 p.350 #49.

(a) $\sum_{k=1}^{10} k = 1+2+3+\cdots+10 = 55$

I used $\sum_{k=1}^{n} k = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$

(b) $\sum_{n=0}^{4} \sin\frac{n\pi}{2} = \underset{0}{\cancel{\sin\frac{0\pi}{2}}} + \underbrace{\sin\frac{\pi}{2}}_{1} + \underset{0}{\cancel{\sin\frac{2\pi}{2}}} + \underbrace{\sin\frac{3\pi}{2}}_{-1} + \underset{0}{\cancel{\sin\frac{4\pi}{2}}} = 0$

47–50. *The accompanying figure shows four regions bounded by the graph of $y = x\sin x$: R_1, R_2, R_3, and R_4, whose areas are 1, $\pi - 1$, $\pi + 1$, and $2\pi - 1$, respectively. (We verify these results later in the text.) Use this information to evaluate the following integrals.*



47. $\int_0^{\pi} x\sin x\,dx$ **48.** $\int_0^{3\pi/2} x\sin x\,dx$

49. $\int_0^{2\pi} x\sin x\,dx$ **50.** $\int_{\pi/2}^{2\pi} x\sin x\,dx$

#47. $\int_0^{\pi} x\sin x\,dx = R_1 + R_2 = 1 + \pi - 1 = \pi$

#48. $\int_0^{3\pi/2} x\sin x\,dx = R_1 + R_2 - R_3 = \pi - (\pi + 1) = -1$

#49. $\int_0^{2\pi} x\sin x\,dx = \underbrace{R_1 + R_2 - R_3}_{-1} - R_4 = -1 - (2\pi - 1) = -2\pi$

#50. $\int_{\pi/2}^{2\pi} x\sin x\,dx = R_2 - R_3 - R_4 = 2\pi - 1$

Check using the Fundamental Theorem of Calculus.

$$\frac{d}{dx}\,x\cos x = 1\cdot\cos x + x\cdot(-\sin x)$$
$$= \cos x - x\sin x$$

$$\frac{d}{dx}(-x\cos x) = -\cos x + x\sin x$$

$$\frac{d}{dx}(-x\cos x + \sin x) = \cancel{-\cos x} + x\sin x + \cancel{\cos x}$$
$$= x\sin x$$

$$\int x\sin x\,dx = -x\cos x + \sin x + C$$

$$\int_0^{\pi} x\sin x\,dx = \left[-x\cos x + \sin x\right]_0^{2\pi}$$

$$= (-2\pi\cdot 1 + 0) - (-0 + 0) = -2\pi.$$

Sec 5.1 p.350 #43.

x	x_0: 0	x_1: 0.5	x_2: 1.0	x_3: 1.5	x_4: 2.0
$f(x)$	5	3	2	1	1

$n=4$

$\Delta x = \frac{2-0}{4} = \frac{1}{2}$

Approximate $\int_0^2 f(x)\,dx$ using left and right Riemann sums with $n=4$ subdivisions.

Left Riemann Sum:

$$\sum_{i=1}^{4} f(x_{i-1})\Delta x = (f(x_0)+f(x_1)+f(x_2)+f(x_3))\Delta x = (5+3+2+1)\times 0.5 = 11\times 0.5 = 5.5.$$

Right Riemann Sum

$$\sum_{i=1}^{4} f(x_i)\Delta x = (f(x_1)+f(x_2)+f(x_3)+f(x_4))\Delta x = (3+2+1+1)\times 0.5 = 7\times 0.5 = 3.5.$$

We don't have enough data to find the midpoint Riemann sum with $n=4$ subdivisions because that would require $f(\frac{x_{i-1}+x_i}{2})$ for $i=1,2,3,4$ i.e. $f(0.25)$, $f(0.75)$, $f(1.25)$, $f(0.75)$.

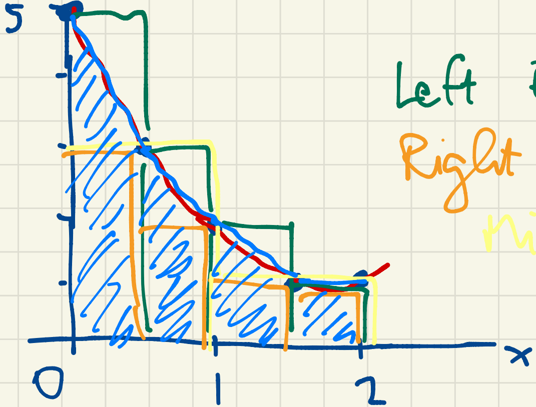
We can give a midpoint Riemann Sum with $n=2$ subdivisions

$\frac{x_0+x_1}{2}=0.5$ $\qquad$ $\frac{x_1+x_2}{2}=1.5$

x	$x_0=0$	0.5	$x_1=1$	1.5	$x_2=2$
$f(x)$		3		1	

$\Delta x = \frac{2-0}{2} = 1$

$$\sum_{i=1}^{2} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x = (f(0.5)+f(1.5))\cdot 1 = 3+1 = 4.$$



Left Riemann Sum

Right Riemann Sum

Midpoint Riemann Sum

Trapezoidal Rule

gives $\frac{5.5+3.5}{2} = 4.5$

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Sec 4.9 p.333 $s'(t) = v(t)$, $v'(t) = a(t) = -g = -9.8\ m/sec^2$

#107. A softball is popped up vertically (from the ground) with a velocity of 30 m/sec.

(a) Find $v(t) = \int a(t)\,dt = \int (-9.8)\,dt = -9.8t + C$, $v(0) = 30 \Rightarrow C = 30.$

$v(t) = 30 - 9.8t$

(b) $s(t) = \int v(t)\,dt = \int (30 - 9.8t)\,dt = 30t - 4.9t^2 + K$, $s(0) = 0 \Rightarrow K = 0$

$s(t) = 30t - 4.9t^2$

(c) Find the time when the ball reaches its highest point. What is the height?

The maximum height occurs when $v(t) = 0 = 30 - 9.8t \Rightarrow t = \frac{30}{9.8} \approx 3.06$ sec

The maximum height is $s(\frac{30}{9.8}) = \frac{30^2}{9.8} - 4.9 \times \frac{30^2}{9.8^2} = \frac{30^2}{2 \times 9.8} = \frac{30^2}{2 \times 9.8} \approx 45.92$ m

(d) The ball strikes the ground when $0 = s(t) = (30 - 4.9t)t \Rightarrow t = \frac{30}{4.9} = 6.12$ sec.

$$\int \frac{dt}{1+t^2} = \tan^{-1} t + C$$

$$\frac{d}{dt}(\tan^{-1} t) = \frac{1}{1+t^2}$$

$$\int \frac{dt}{1+4t^2} = \frac{1}{2}\tan^{-1}(2t) + C$$

$$\frac{d}{dt}\left(\tan^{-1}(2t)\right) = \frac{1}{1+(2t)^2} \cdot 2 = \frac{2}{1+4t^2}$$

Divide both sides by 2.

$$\frac{d}{dt} \frac{1}{2}\tan^{-1}(2t) = \frac{1}{1+4t^2}$$

$$\int \frac{dt}{4+t^2} = \frac{1}{4}\int \frac{dt}{1+\frac{t^2}{4}} = \frac{1}{2}\tan^{-1}\frac{t}{2} + C$$

$$\frac{d}{dt}\tan^{-1}\frac{t}{2} = \frac{1}{1+(\frac{t}{2})^2} \cdot \frac{1}{2}$$

$$\frac{d}{dt}\frac{1}{2}\tan^{-1}\frac{t}{2} = \frac{1}{4} \cdot \frac{1}{1+\frac{t^2}{4}} = \frac{1}{4+t^2}$$

Alternatively, substitute (Sec 5.5)

$$\int \frac{dt}{1+4t^2} = \int \frac{du/2}{1+u^2} = \frac{1}{2}\int \frac{du}{1+u^2}$$

Let $u = 2t$ $\left(\frac{du}{dt} = 2\right)$

$du = 2\,dt$

$dt = \frac{du}{2}$

$$= \frac{1}{2}\tan^{-1} u$$

$$= \frac{1}{2}\tan^{-1}(2t) + C.$$

$$\int \frac{dt}{4+t^2} = \int \frac{2du}{4+4u^2} = \frac{1}{2}\int \frac{du}{1+u^2}$$

$t = 2u$

$dt = 2\,du$

$\frac{dt}{du} = 2$

$$= \frac{1}{2}\tan^{-1} u = \frac{1}{2}\tan^{-1}\frac{t}{2} + C$$

$$\int \frac{dt}{4+t^2} = \int \frac{2\sec^2\theta\, d\theta}{4+4\tan^2\theta}$$

$$= \int \frac{2\sec^2\theta\, d\theta}{4\sec^2\theta}$$

$$= \int \frac{1}{2}\, d\theta$$

$$= \frac{1}{2}\theta$$

$$= \frac{1}{2}\tan^{-1}\frac{t}{2} + C$$

$t = 2\tan\theta$

$1+\tan^2\theta = \sec^2\theta$

$\frac{dt}{d\theta} = 2\sec^2\theta$

$\Rightarrow \frac{t}{2} = \tan\theta$

$\tan^{-1}\frac{t}{2} = \theta$

$$\int \frac{t\,dt}{1+t^2} = \int \frac{\frac{1}{2}\,du}{1+u} = \frac{1}{2}\int \frac{du}{1+u} = \frac{1}{2}\ln(1+u) = \frac{1}{2}\ln(1+t^2) + C$$

Let $u = t^2$

$$\frac{du}{dt} = 2t$$

$$du = 2t\,dt$$

$$\frac{1}{2}\,du = t\,dt$$

$$\frac{d}{dt}\tan(t^2) = \frac{1}{1+t^4}\cdot 2t = \frac{2t}{1+t^4}$$

$$\frac{d}{dt}\left(\frac{1}{1+t}\right) = \frac{d}{dt}(1+t)^{-1} = -(1+t)^{-2}\cdot 1 = -\frac{1}{(1+t)^2}$$

$$\frac{d}{du}\left(\ln(1+u)\right) = \frac{1}{1+u}\cdot 1 = \frac{1}{1+u}$$

$$\frac{d}{du}(\ln u) = \frac{1}{u}$$

Check: $\frac{d}{dt}\,\frac{1}{2}\ln(1+t^2) = \frac{1}{2}\cdot\frac{1}{1+t^2}\cdot 2t = \frac{t}{1+t^2}.$