

Putnam Team Seminar

Practice Problems 2

Monday, September 8, 2025

1. Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?
2. Let f and g be (real-valued) functions defined on an open interval containing 0, with g nonzero and continuous at 0. If fg and f/g are differentiable at 0, must f be differentiable at 0?

3. Find the volume of the region of points (x, y, z) such that

$$(x^2 + y^2 + z^2 + 8)^2 \leq 36(x^2 + y^2).$$

4. Evaluate

$$\sqrt[8]{2207 - \frac{1}{2207 - \frac{1}{2207 - \dots}}}.$$

Express your answer in the form $\frac{a+b\sqrt{c}}{d}$ where a, b, c, d are integers.

5. Prove

$$\frac{22}{7} - \pi = \int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx.$$

6. Determine all polynomials $P(x)$ such that $P(x^2+1) = P(x)^2 + 1$ and $P(0) = 0$.

7. Let $f_0(x) = e^x$ and $f_{n+1}(x) = xf'_n(x)$ for $n = 0, 1, 2, \dots$. Show that $\sum_{n=0}^{\infty} \frac{f_n(1)}{n!} = e^e$.