Practice Problems 7

Monday, October 20, 2025

1. Prove that $\binom{pa}{pb} \equiv \binom{a}{b} \pmod{p}$ for all integers p, a, b with p prime and $a \geqslant b \geqslant 0$.

2. (a) If every point of the plane is painted one of three colors, do there necessarily exist two points of the same color exactly one inch apart?

   (b) What if 'three' is replaced by 'nine'?

3. Find all integral solutions of the equation $|p^r - q^s| = 1$ where p are q are prime numbers and r and s are positive integers larger than unity. Prove that there are no other solutions.

4. A well known theorem asserts that a prime $p > 2$ can be written as the sum of two perfect squares ($p = m^2 + n^2$, with m and n integers) if and only if $p \equiv 1 \mod 4$. *Assuming* this result, find which primes $p > 2$ can be written in each of the following forms, using (not necessarily positive) integers x and y:

   (a) $x^2 + 16y^2$;

   (b) $4x^2 + 4xy + 5y^2$.

5. Let $\mathcal{C}$ be the class of all real-valued continuously differentiable functions f on the interval $0 \leqslant x \leqslant 1$ with $f(0) = 0$ and $f(1) = 1$. Determine the largest real number u such that

$$u \leqslant \int_0^1 |f'(x) - f(x)|\, dx$$

   for all f in $\mathcal{C}$.

6. Is there an infinite sequence of real numbers $a_1, a_2, a_3, \ldots$ such that

$$a_1^m + a_2^m + a_3^m + \cdots = m$$

   for every positive integer m?

7. Evaluate $\displaystyle \lim_{n\to\infty} \frac{1}{n^4} \prod_{i=1}^{2n} \left(n^2 + i^2\right)^{1/n}$.