HW2

Due 5:00 pm, Wednesday, March 4, 2026 on WyoCourses. March forth and conquer!

> *Instructions:* See the syllabus for general instructions for completing homework. Further details are found at the FAQ page linked from the syllabus. Always check your answers wherever feasible. Write clearly, using correct notation.

1. *(12 points)* A subgroup $G \leqslant S_8$ has elements which we partially list as shown:

$$(),\ (1234)(5678),\ (1537)(2846),\ (1432)(5876),\ \underline{\quad},\ \underline{\quad},\ \underline{\quad},\ \underline{\quad}.$$

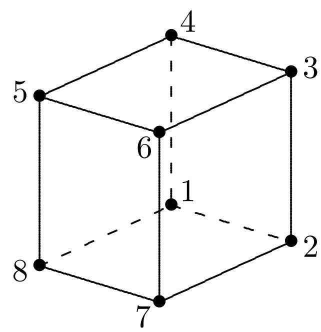
   (a) Find explicitly the four missing elements of G.

   (b) How many elements of each order does G have?

   (c) Is G abelian?

2. *(20 points)* How many elements of each order does S_6 have? (*Do not* list them all.)

3. *(24 points)* The vertices of a cube are labelled 1,2,3,4,5,6,7,8 along a Hamilton path as shown:



(By a *Hamilton path*, we mean that 12, 23, 34, . . . , 78 are edges of the cube. So is 81, thereby completing a *Hamilton cycle*.) Let G be the symmetry group of the cube, and let $H \leqslant G$ be the group of rotational symmetries, so that $|G| = 48$ and $|H| = 24$. It is convenient to represent G and H as permutation groups on the 8 vertices, so that they can be studied as subgroups of S_8. Note that all 24 elements of H are orientation-preserving. Half of the elements of G (those not in H) are orientation-reversing (these include the reflections). Thus, for example, $(248)(357) \in H$ is a $120°$ rotation about the axis joining opposite vertices 1 and 6; $(14)(23)(58)(67) \in G$ is a reflection in a 'horizontal' plane of symmetry midway between the opposite faces 1278 and 4365. The symmetry $(123658)(47) \in G$ is neither a rotation nor a reflection. The element $(16)(25)(38)(47) \in G$ (often called an *inversion*) is special, as it commutes with every element of G. Note that every rotational symmetry of the cube has an axis, this being a line through the center of the cube, each of whose points is fixed by the symmetry. A reflective symmetry of the cube, by contrast, fixes every point of some plane through the center of the cube; and this plane acts as a 'mirror' reflecting points of the cube to points on the other side.

For each of the following types of elements of G, count all elements of the given type, *and* list them explicitly:

(a) $0°$ rotation(s) in H. Such elements have order 1.

(b) $90°$ rotations in H. Such elements have order 4.

(c) $120°$ rotations in H. Such elements have order 3.

(d) $180°$ rotations in H. These elements have order 2.

(e) reflections in G. These elements have order 2.

(f) all other elements in G. What are the orders of these elements?

For each prime p and each positive integer n; denote by $GL_n(p) = GL_n(\mathbb{F}_p)$ the group of all invertible $n \times n$ matrices with entries in the field $\mathbb{F}_p = \{0, 1, 2, \ldots, p-1\}$ (the field of order p, in which addition and multiplication are performed mod p).

4. *(20 points)* Let $G = GL_2(\mathbb{F}_2)$.
   (a) How many elements of each order does G have? List them explicitly.
   (b) Write down an explicit isomorphism from S_3 to G.

5. *(24 points)* Let $G \leqslant GL_3(\mathbb{F}_p)$ be the subgroup consisting of all matrices of the form

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

   where $a, b, c \in \mathbb{F}_p$. Then G is a subgroup of order p^3.
   (a) If $p = 2$, then G is a group of order 8. How many elements of each order does G have?
   (b) If $p = 3$, then G is a group of order 27. How many elements of each order does G have? (You *do not* need to list them all.)