HW3

Due 5:00 pm, Friday, April 17, 2026 on WyoCourses.

Instructions: See the syllabus for general instructions for completing homework. *Do yourself a very big favor by doing your own work.* (This should go without saying; but some students had to learn this lesson the hard way on our midterm test.) Further details are found at the FAQ page linked from the syllabus. Always check your answers wherever feasible. Write clearly, using correct notation.

In #1 and #2, we consider the product of all the elements of a finite multiplicative group G. This is not a well-defined concept in general, since the order in which we list the elements in this product will affect the outcome. For example in the dihedral group of order 8, we have

$$I \cdot R \cdot R^2 \cdot R^3 \cdot D \cdot D' \cdot H \cdot V = R^2,$$

whereas

$$R \cdot R^3 \cdot V \cdot I \cdot D' \cdot R^2 \cdot H \cdot D = I;$$

so in this group, the 'product of all elements of the group' is not a well-defined element. But in an abelian group, the product is obviously well-defined.

1. *(20 points)* What are the possible values obtained as 'the product' of all the elements of S_3?

2. *(20 points)* What is the product of all the elements of $C_n = \{1, g, g^2, \ldots, g^{n-1}\}$, the cyclic group of order n? (Of course the answer cannot depend on the order in which you multiply the elements of the group; but the answer *does* depend on n.)

In class, I will show that the number of homomorphisms from C_n to C_n is exactly n. There are exactly 8 homomorphisms from a Klein four-group to itself.

3. *(20 points)* How many homomorphisms are there from S_3 to S_3? Describe them in the simplest way that you can. (*Hint*: The number of homomorphisms $S_3 \to S_3$ is more than 2 but less than 100.)

Assuming G and H are multiplicative groups, a bijection $f : G \to H$ is an *anti-isomorphism* if $f(ab) = f(b)f(a)$ for all $a, b \in G$.

4. *(20 points)* Let G, H, K be groups, with identity elements 1_G, 1_H, 1_K respectively; and suppose that $\phi : G \to H$ and $\psi : H \to K$ are anti-isomorphisms.
   (a) Show that $\phi(1_G) = 1_H$.
   (b) Show that $\psi \circ \phi : G \to K$ is an isomomorphism.

5. *(20 points)* Let $G = SL_2(\mathbb{R}) = \left\{ \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] : a, b, c, d \in \mathbb{R},\ ad - bc = 1 \right\}$. Define two maps $G \to G$ by $\phi(A) = A^{-1}$ and $\psi(A) = A^T$. Here A^T is the transpose of A, i.e. $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]^T = \left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right]$. Both the maps ϕ and ψ are anti-isomorphisms, a fact that is well-known and which you may assume. Moreover, the fact that $\phi \circ \psi = \psi \circ \phi$ is easily verified (you should check this yourself; but you do not need to show me a formal proof). So $(A^{-1})^T = (A^T)^{-1}$; and this result is usually written simply as A^{-T}. By #4(b), the map $A \mapsto A^{-T}$ is an automorphism of G. Find an element $B \in G$ such that $A^{-T} = BAB^{-1}$ for all $A \in G$.

6. *(20 points)* Let p be a prime, n a positive integer, and consider the group $G = GL_n(F)$ where $F = \mathbb{F}_p = \{0, 1, 2, \ldots, p-1\}$, the field of order p (the integers mod p). Then G permutes the vectors of the n-dimensional vector space F^n. That is, for each $A \in G$, the map $F^n \to F^n$, $v \mapsto Av$ is a permutation of the p^n vectors. This permutation is either even or odd. For example, the identity matrix $I \in G$ gives the identity permutation $v \mapsto v$, which is an even permutation. Find an explicit example in which $A \in G$ gives an odd permutation of the vectors. (Please be explicit! Obviously you must give an explicit choice of p and n, before giving an explicit matrix $A \in G$ and an explanation why the resulting permutation of vectors is odd.)