Sample Test

This sample test is intended as preparation for the Test (Monday, March 30, 2026 during class time) in approximate difficulty and style, but not length: the actual test will consist of 4 questions similar to #1–#11 below, each worth about 20 points; and a ten-part question similar to #12 below, worth 30 points. The grade will be recorded out of 100 points (since there will be bonus points available). The actual content of the test will be restricted to material covered in class prior since the beginning of the semester, until Friday, March 6, inclusive.

Instructions. The only aids allowed are a hand-held calculator and one 'cheat sheet', i.e. an 8.5″ ×11″ sheet with information written on one side in your own handwriting. No cell phones are permitted (in particular, a cell phone may not be used as a calculator). Answer as clearly and precisely as possible. *Clarity is required for full credit!* Time permitted: 50 minutes.

1. Find a subgroup $G \leqslant S_4$ isomorphic to the dihedral group of order 8. Answer by exhibiting G as a set of 8 explicit permutations (a Cayley table is not necessary).

2. Let S be the set of all 2×2 real matrices of the form
$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
where $a \in \mathbb{R}$. Is every invertible 2×2 real matrix expressible as a product of elements of S? Justify your answer.

3. Let H_1, H_2 be subgroups of a multiplicative group G. Give an example to show that the subset defined by
$$H_1H_2 = \{h_1h_2 \,:\, h_1 \in H_1,\ h_2 \in H_2\} \subseteq G$$
is not necessarily a subgroup.

4. There is an isomorphism $f : S_6 \to S_6$ satisfying $f((12)) = (12)(34)(56)$ and $f((23)) = (13)(25)(46)$. (This is a not-so-obvious fact that you should assume!)
   (a) Using the information given, evaluate $f((123))$.
   (b) How many transpositions does S_6 have?
   (c) A *triple transposition* in S_6 is a product of three disjoint transpositions. (For example, $f((12)) = (12)(34)(56)$ is a triple transposition.) How many triple transpositions does S_6 have?

5. Give an example of two finite groups G and H of the same order, and having the same number of elements of each order, such that $G \not\cong H$.

6. Let G be a multiplicative group containing elements g, h of order $|g| = 3$ and $|h|=5$.
   (a) What can be said about $|gh|$ based on the information given?
   (b) What can be said about $|gh|$ if g commutes with h?

7. Find two elements $\sigma, \tau \in S_5$ such that $\langle \sigma, \tau \rangle = S_5$, thus showing that S_5 is generated by two of its elements. (Just state your choices for $\sigma, \tau \in S_5$; you are *not* required to prove that they generate S_5.)

8. Find the order of the group $G = GL_2(\mathbb{F}_5)$, the group of invertible matrices over the field $\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$ (the integers mod 5).

9. How many elements in the group $G = GL_2(\mathbb{F}_5)$ commute with *every* element of G? List them explicitly.

10. Each of the following objects (in 2 or 3 dimensions) has a symmetry group G. In each case, determine the order $|G|$; also determine whether or not G is abelian (e.g. abelian of order 3, infinite nonabelian, etc.).
   (a) a regular pentagonal prism (whose cross sections are regular pentagons) 
   (b) a brick of size $2'' \times 3'' \times 8''$ 
   (c) the letter E
   (d) the string of letters S O S
   (e) the string of letters Y O U
   (f) the infinite string of letters $\cdots$ E E E E E E E E E E E $\cdots$
   (g) the infinite string of letters $\cdots$ H H H H H H H H H H $\cdots$
   (h) the infinite string of letters $\cdots$ H H H H E T H H H H $\cdots$

11. Consider two lines ℓ and ℓ' in the plane $\mathbb{R}^2$, where ℓ is the x-axis (i.e. the line $y = 0$), and ℓ' is the line through the origin with slope 1 (i.e. the line $y = x$). Let $S, S' : \mathbb{R}^2 \to \mathbb{R}^2$ be the bijections defined by reflecting in the lines ℓ and ℓ' respectively. How do the composite maps SS' and $S'S$ act on $\mathbb{R}^2$? Describe each of these transformations using the standard geometric terminology. (As usual, $SS' = S \circ S'$ and $S'S = S' \circ S$.)

12. Answer TRUE or FALSE to each of the following statements.

(a) For every positive integer n, there is a group of order n. _____*(True/False)*

(b) If two groups are isomorphic, then they necessarily have the same order. _____*(True/False)*

(c) There exist infinitely many groups of order 24, no two of which are isomorphic. _____*(True/False)*

(d) If G is a finite group, then every element of G has finite order._____*(True/False)*

(e) There exist physical objects with nontrivial rotational symmetry, but no reflective symmetry. _____*(True/False)*

(f) The multiplicative group $\mathbb{C}^\times$ of nonzero complex numbers has elements of all possible orders, finite and infinite. _____*(True/False)*

(g) The additive group of integers $\mathbb{Z}$ is isomorphic to the additive group of even integers $2\mathbb{Z} = \{2k : k \in \mathbb{Z}\}$. _____*(True/False)*

(h) If G is a group containing two elements g, h of finite order, then their product gh has finite order. _____*(True/False)*

(i) Let G be a set with two binary operations '$*$' and '$\circ$', such that $g \circ f = f * g$ for all $f, g \in G$. If G is a group under the operation '$*$', then G is also a group under the operation '$\circ$'. _____*(True/False)*

(j) Given a finite group G of order n, there is a binary operation '$*$' on the set $[n] = \{1, 2, \ldots, n\}$ which defines a group isomorphic to G. _____*(True/False)*