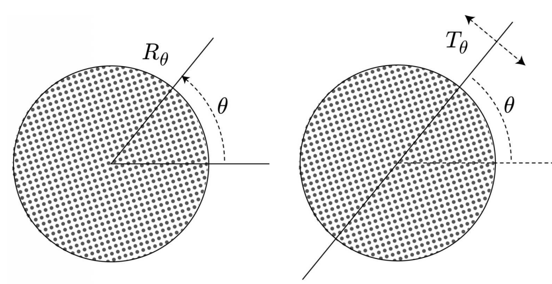
Final Practice and Sample Problems

This list of problems is intended as practice in preparation for the Final Examination (10:15 am–12:15 pm on Friday, May 15, 2026 in our usual lecture room, BU 24). You should expect the actual exam have about eight problems (somewhat fewer than you see in this list of practice problems). The last problem will consist of ten True/False statements. Some bonus points will be available. The actual content will consist of all material covered in class this semester, and all related handouts. Somewhat greater weight will be placed on the later material (covered after Spring Break).

Instructions. The only aids allowed are a hand-held calculator and one 'cheat sheet', i.e. an 8.5″×11″ sheet with information written on one side in your own handwriting. No cell phones are permitted (in particular, a cell phone may not be used as a calculator). Answer as clearly and precisely as possible. *Clarity is required for full credit!* Time permitted: 120 minutes.

1. Let f and g be homomorphisms from a group G to a group H. Define $S = \{x \in G : f(x) = g(x)\}$. Is S necessarily a subgroup of G? Either prove this, or give a counterexample.

2. Consider the closed disk $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$, and let G be the symmetry group of D. For each angle θ, denote by $R_\theta \in G$ the rotational symmetry of D by an angle θ counterclockwise about its center. Also denote by $T_\theta \in G$ the reflective symmetry of D whose axis has angle θ relative to the positive x-axis as shown.



   (a) Show that every rotational symmetry $R_\theta \in G$ is a product of two reflective symmetries, i.e. $R_\theta = T_\alpha T_\beta$ for some angles α, β depending on θ.

   (b) Show that any two reflective symmetries $T_\alpha, T_\beta \in G$ are conjugate in G; in fact, show that $T_\beta = R_\theta T_\alpha R_\theta^{-1}$ for some angle θ depending on α and β.

3. Let $G = GL_2(\mathbb{R})$ and define $\theta : GL_2(\mathbb{R}) \to GL_2(\mathbb{R})$ by $\theta(A) = \frac{1}{\det A} A$ for every matrix $A \in G$. (For example, $\theta\left(\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}\right) = \frac{1}{4}\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 0.25 & 0.5 \\ 0 & 1 \end{bmatrix}$.) Is θ an automorphism of G? Justify your answer.

4. Consider the symmetric group $G = S_7$ and the centralizer of a transposition

$$H = C_G\big((67)\big) = \{\sigma \in G \,:\, \sigma(67) = (67)\sigma\}.$$

   (a) Determine $|H|$.

   (b) Identify H (up to isomorphism) using terminology or notation for the groups we have studied in class this semester.

5. Let $G = GL_3(\mathbb{R})$, the group of all invertible real 3×3 matrices.

   (a) Does G have a subgroup isomorphic to $GL_2(\mathbb{R})$? Justify your answer.

   (b) Does G have a subgroup of order 5? Justify your answer.

   (c) Does G have an infinite abelian subgroup? Justify your answer.

6. Let G be a cyclic group of order 100.

   (a) How many elements of each order does G have?

   (b) How many homomorphisms are there from G to G?

   (c) How many automorphisms does G have?

   (d) How many elements $h \in G$ are there satisfying $\langle h \rangle = G$?

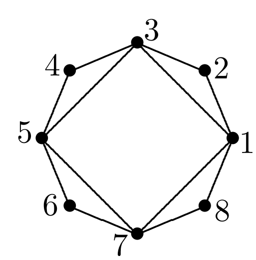
   (e) If $g \in G$ has order 4, how many elements $h \in G$ are there satisfying $\langle g, h \rangle = G$?

7. In A_6, how many elements of order 4 are there? How many conjugacy classes of such elements are there? Do the same for elements of order 5.

8. How many *abelian* groups of order 100 are there (up to isomorphism)? List them, using terminology or notation we have used in class this semester.

9. Let $H \leqslant S_8$ be the subgroup generated by all the 7-cycles, i.e. the set of all elements in S_8 conjugate to (1234567). Identify H as a known group using standard terminology we have used in class this semester. Justify your answer.

10. Identify the group G consisting of all automorphisms of the graph shown on the right. You should indicate the usual name for G, the order of G, and express G an explicit subgroup of S_8. You should in fact be able to express $G = \langle \alpha, \beta \rangle$ for an explicit pair of permutations $\alpha, \beta \in S_8$.



11. Answer TRUE or FALSE to each of the following statements.

(a) Every finite group is isomorphic to a subgroup of S_n for some $n \geq 1$.
_____*(True/False)*

(b) Every nontrivial finite group contains an element of prime order.
_____*(True/False)*

(c) If x and y are two elements in a group G, then xy and yx are necessarily conjugate in G. _____*(True/False)*

(d) If G and H are groups, then every subgroup of $G \times H$ has the form $A \times B$ for some subgroup $A \leqslant G$, and some subgroup $B \leqslant H$. _____*(True/False)*

(e) If G is an abelian group, then any two automorphisms of G commute with each other. _____*(True/False)*

(f) If $x, y \in G$ are two group elements of the same order, then they are necessarily conjugate in G. _____*(True/False)*

(g) If H and K are subgroups of a group G, then so is their product $HK = \{hk : h \in H,\ k \in K\}$. _____*(True/False)*

(h) If g, h are elements of a group G, then there exists a homomorphism $\phi : G \to G$ such that $\phi(g) = h$. _____*(True/False)*

(i) If G is a group of order n and $\gcd(k, n) = 1$, then the k-th power $\theta_k : G \to G$, $\theta_k(x) = x^k$ is necessarily bijective. _____*(True/False)*

(j) If G and H are finite groups having the same number of elements of each order, then necessarily G and H are isomorphic. _____*(True/False)*