Solutions to HW1

1. (a) 2 (b) Yes (c) 2 (d) 0
2. (a) 4 (b) Yes (c) 2 (d) 2
3. (a) 2 (b) Yes (c) 1 (d) 1
4. (a) 1 (b) Yes (c) 1 (d) 0

5. Evidently b is the identity. The Cayley table of G becomes

	b	a	c	e	d
b	b	a	c	e	d
a	a	c			
c	c		d		
e	e				
d	d				

Using the 'Latin square condition' (the condition that every group element appears exactly once in each row and column), the locations of the missing two 'd' entries are forced:

	b	a	c	e	d
b	b	a	c	e	d
a	a	c		d	
c	c		d		
e	e	d			
d	d				

The Latin square condition yields $ac \in \{b, e\}$. However, if $ac = b$ then $c = a^{-1}$ and $ca = b$; and then it is easy to see that there is no way to complete the Cayley table satisfying the Latin square condition. So we must instead have $ac = e$ and $ad = b$. This gives $d = a^{-1}$, so $da = b$. Using the Latin square condition, we find a unique way to complete the Cayley table:

	b	a	c	e	d
b	b	a	c	e	d
a	a	c	e	d	b
c	c	e	d	b	a
e	e	d	b	a	c
d	d	b	a	c	e

The familiar cyclic pattern in the rows of the Cayley table shows us that G is a cyclic group of order 5. Our judicious choice of ordering of the elements of G shows that

$G = \langle a \rangle = \{1, a, a^2, a^3, a^4\}$. If we had listed the elements of G in alphabetical order, it would not have been evident (from our Cayley table) that G was a group at all, let alone cyclic. With different choices of ordering of the elements of G, the resulting Cayley table could have revealed $G = \langle c \rangle$, or $G = \langle d \rangle$, or $G = \langle e \rangle$. Note that G has one element b of order 1, and four elements a, c, d, e of order 5. Since G is cyclic, G is abelian. (This fact is also evident from the fact that the Cayley table is symmetric.)

6. More than enough examples were presented in class. Recall that we produced rotation matrices in $GL_2(\mathbb{R})$ of several orders. These examples actually lie in $SL_2(\mathbb{R})$, as every rotation has determinant 1. Note that if R has order 6, then R^2 and R^4 have order 3, and R^3 has order 2.

   (a) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

   (b) Yes, G has a unique element of order 2. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G$ has order 2, then $A^{-1} = A$. Since $ad - bc = 1$, this says that $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and so $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ with $a = \pm 1$. The unique element of order 2 is therefore $-I$.

   (c) $\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$

   (d) No, G has other elements of order 3 such as $\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$, $\frac{1}{2}\begin{bmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{bmatrix}$, and infinitely many other examples.

   (e) There are infinitely many examples, including $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}$, $\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$, $\ldots$.