Solutions to Sample Test

1. One solution is $G = \{(),\ (1234),\ (13)(24),\ (1432),\ (13),\ (24),\ (12)(34),\ (14)(23)\}$ obtained by considering the permutation action of G on the four vertices of a square, which we have chosen to label as 1,2,3,4 in counter-clockwise order.

   In fact S_4 has exactly three dihedral subgroups of order 8. In addition to the example above, there are subgroups
   $$\{(),\ (1324),\ (12)(34),\ (1423),\ (12),\ (34),\ (13)(24),\ (14)(23)\}$$
   and
   $$\{(),\ (1243),\ (14)(23),\ (1342),\ (14),\ (23),\ (12)(34),\ (13)(24)\}$$
   arising from alternative labelings of the four vertices of the square (1,3,2,4 or 1,2,4,3 in counter-clockwise order, respectively).

2. No. All elements of S have determinant ± 1; so every product of elements of S has determinant ± 1. No element of $GL_2(\mathbb{R})$ having any other determinant than 1 or -1, can be expressed as a product of elements of S. (In fact $G = \{A \in GL_2(\mathbb{R}) : \det A = \pm 1\}$ is a subgroup of $GL_2(\mathbb{R})$; and it may be shown that $\langle S \rangle = G$, i.e. the matrices expressible as products of elements of S are precisely the elements of G.)

3. In S_3 we have $\langle(12)\rangle\langle(13)\rangle = \{(),\ (12)\}\{(),\ (13)\} = \{(),\ (12),\ (13),\ (132)\}$ which is not a subgroup: it is not closed since it does not contain the product $(13)(12) = (123)$.

4. (a) $f((123)) = f((12)(23)) = f((12))f((23)) = (12)(34)(56)(13)(25)(46) = (145)(263)$.
   (b) S_6 has $\binom{6}{2} = 15$ transpositions.
   (c) S_6 has $\frac{1}{6}\binom{6}{2}\binom{4}{2}\binom{2}{2} = \frac{15\cdot 6\cdot 1}{6} = 15$ triple transpositions. Later, we will see an explanation for why the answers to (b) and (c) coincide. This will also explain other similar apparent coincidences observed in class, e.g. when counting the two cycle structures of elements of order 3 in S_6.

5. The smallest example comes from HW2 #5(b), where we encountered a nonabelian group G of order 27 having 26 elements of order 3. There is of course also an abelian group of order 27 having 26 elements of order 3, namely the additive group of the vector space $\mathbb{F}_3^3$. These two groups cannot be isomorphic because one is abelian, while the other is not.

6. (a) Almost nothing can be said about $|gh|$ in general. Clearly $|gh| \neq 1$; otherwise g and h would be inverses of each other, which is not possible since they do not have

the same order. But from the information given, $|gh|$ could be any integer $\geqslant 2$, or even infinite. Consider these examples in S_8:

$$\begin{array}{ll} |(123)(45678)| = 15 & |(123)(12345)| = |(13452)| = 5 \\ |(123)(34567)| = |(1234567)| = 7 & |(123)(14352)| = |(14)(35)| = 2 \\ |(123)(23456)| = |(12)(3456)| = 4 & |(123)(13245)| = |(245)| = 3 \end{array}$$

(b) If g and h commute, then $|gh| = 15$. To see this, note that $(gh)^k = g^k h^k$ and in particular $(gh)^{15} = g^{15}h^{15} = 1$; but $g^k h^k \neq 1$ for $k \in \{1, 2, 3, \ldots, 14\}$ since $g^k \neq h^{-k}$ (the left side has order 1 or 3, but the right side has order 1 or 5).

7. There are many possible choices of the pair $\{\sigma, \tau\}$, but be sure that at least one of them is an odd permutation. A reasonable choice is $\sigma = (12345)$, $\tau = (12)$. To see that $\langle \sigma, \tau \rangle = S_5$ in this case, it suffices to show that every transposition is contained in $\langle \sigma, \tau \rangle$, since (as pointed out in class) S_n is generated by its transpositions. Indeed, the transpositions $(23), (34), (45), (15)$ are obtained as $\sigma^j \tau \sigma^{-j}$ for $j = 1, 2, 3, 4$ respectively. From these, one obtains $(13) = (12)(23)(12)$. Finally, the remaining transpositions $(24), (35), (14), (25)$ are obtained as $\sigma^j (13) \sigma^{-j}$ for $j = 1, 2, 3, 4$ respectively.

8. $|G| = 24 \cdot 20 = 480$. Here we note that $\mathbb{F}_5^2$ has 25 vectors. To form an invertible matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in G$, there are 24 choices for the first column $\left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right]$ (any nonzero vector will work); then $25 - 5 = 20$ choices for the second column $\left[\begin{smallmatrix} b \\ d \end{smallmatrix}\right]$ (any column that is not a scalar multiple of the first column will work).

9. Exactly four elements of G commute with all elements of G. These are the matrices $aI = \left[\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right]$ for nonzero scalar values $a \in \{1, 2, 3, 4\}$. Given a matrix $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in G$ which commutes with all elements of G, then in particular
$$A\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]A \quad \text{and} \quad A\left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]A.$$
This gives us eight linear equations in the four unknowns $a, b, c, d \in \mathbb{F}_5$, yielding $a = d$, $b = c = 0$. This gives $A = \left[\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right] = aI$. We must also have $a \neq 0$ in order that $A \in G$. Conversely, every matrix of the form $A = aI$ with $a \neq 0$ commutes with every element of G. This proves our claim that an element $A \in G$ commutes with every element of G *iff* $A = aI$ for some $a \in \{1, 2, 3, 4\}$.

10. (a) G is nonabelian of order 20. The prism has 10 corners; and for any two corners, G has exactly two isometries taking one to the other (one of which preserves orientation, and the other reverses orientation). There is one 'horizontal' plane of symmetry; also five vertical planes of symmetry, in planes spaced 36° apart. The reflection in the horizontal plane of symmetry commutes with any of the reflections in vertical planes of symmetry; but two of the reflections in vertical planes of symmetry do not commute.

(b) G is abelian of order 8, generated by the reflections in the three planes of symmetry. All seven non-identity elements of G have order 2. This includes the three half-turns (180° rotations) about the three axes of symmetry; and the inversion in the center (mapping each vector to its negative, assuming we place the origin of our coordinate system at the center of the brick).

(c) G is abelian of order 2 (a cyclic group generated by the reflection in the horizontal line of symmetry).

(d) G is abelian of order 2 (a cyclic group generated by the half-turn, i.e. 180° rotation, about the center of the O).

(e) G is abelian of order 1. It is the trivial group, since the string of letters has no nontrivial symmetries.

(f) G is infinite abelian. It is generated by the translation one unit to the right (a unit being the distance between the centers of two adjacent E's; and the reflection in the horizontal axis of symmetry. The two generators commute with each other.

(g) G is infinite nonabelian. Let ℓ_1 be the horizontal line of symmetry; for ℓ_2 take the vertical line of symmetry half-way between two adjacent H's; and let ℓ_3 be the vertical line of symmetry through the middle of the H immediately to the right of ℓ_2. The reflections R_1, R_2, R_3 in the axes ℓ_1, ℓ_2, ℓ_3 respectively, are generators for G. Note that R_2R_3 is a shift one unit to the left, whereas $R_3R_2 = (R_2R_3)^{-1}$ is a shift one unit to the right, so G is nonabelian.

(h) G is abelian of order 1. It is the trivial group, since the string of letters has no nontrivial symmetries.

11. SS' is a 90° rotation about the origin, in the clockwise direction. Its inverse is $S'S$, which is a 90° rotation about the origin, in the counter-clockwise direction.

12. (a) T (b) T (c) F (d) T (e) T (f) T (g) T (h) F (i) T (j) T

Here are some remarks and partial explanations for answers in #12:

(a) For every positive integer n, there is a cyclic group of order n.

(b) Every group isomorphism is in particular a bijection.

(c) For each n, there are only finitely many groups of order n up to isomorphism, since there are only finitely many ways to complete a Cayley table with the symbols $1, 2, \ldots, n$. (By the way, there are exactly 15 different groups of order 24 up to isomorphism.)

(d) If G is a finite group, and $g \in G$, then the elements $1, g, g^2, g^3, \ldots \in G$ cannot all be distinct; so $g^i = g^j$ for some $i < j$, in which case $g^{j-i} = 1$ and $|g| \leqslant j - i$.

(e) Here are two objects, each with 180° degree rotational symmetry, but no reflective symmetry:

(f) The multiplicative group $\mathbb{C}^\times$ has an element 2 of infinite order; and for each positive integer n, the element $e^{2\pi i/n}$ has order n.

(g) The map $x \mapsto 2x$ is an isomorphism from the additive group of integers $\mathbb{Z}$ to the additive group of even integers $2\mathbb{Z} = \{2k : k \in \mathbb{Z}\}$.

(h) In the group G of transformations of the Euclidean plane, take g to be a reflection in the y-axis, and h to be a reflection in the line $x = 1$; then gh is a translation 2 units to the left; and in particular, h has infinite order.

(i) It is easy to verify the group axioms for the second binary operation. Note that for $g \in G$, the inverse of g with respect to '$\circ$' is the same as the inverse of g with respect to '$*$'; and denoting this inverse by g^{-1}, the map $g \mapsto g^{-1}$ is an isomorphism from one group to the other.

(j) It is a simple matter to rename the elements of G as $1, 2, 3, \ldots, n$. (More precisely, if $|G| = n$, then we have a bijection $G \to \{1, 2, \ldots, n\}$. Renaming the elements of G in this way, we obtain an isomorphism from G to a new group with binary operation $*$ such that this renaming is in fact a group isomorphism.