Solutions to the Test

1. (a) The clcokwise 90° rotation about the origin is represented by the matrix $g = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, giving a cyclic subgroup $\langle g \rangle = \{I, g, g^2, g^3\} = \{\pm I, \pm g\} < G$ of order 4.

   (b) $\left\langle \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle$ represents the group generated by the reflections in the two coordinate axes (e.g. the symmetry group of a rectangle). This is a Klein four-group (a non-cyclic group of order 4).

   (c) $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ are two examples of elements of infinite order. In fact, almost every element of G has infinite order.

2. (a) $|S_5| = 5! = 120$.

   (b) $|\sigma| = 2$.

   (c) $|\tau| = 5$.

   (d) $\sigma^{-1} = \sigma = (12)(34)$.

   (e) $\tau^{-1} = (15432)$.

   (f) There are exactly 8 elements of S_5 commuting with σ: (), (12)(34), (12), (34), (13)(24), (14)(23), (1324), (1423). These elements form a dihedral group of order 8 (see the solutions to #1 in the Sample Test).

   (g) There are exactly 5 elements of S_5 commuting with τ, namely the powers of τ: (), (12345), (13524), (14253), (15432).

   (h) $\sigma^7\tau^7 = \sigma\tau^2 = (12)(34)(13524) = (14235)$.

   (i) $\tau = (15)(14)(13)(12)$.

   (j) There are $5{\cdot}3 = 15$ elements $(ij)(k\ell) \in S_5$ having the same cycle structure as σ: there are $\binom{5}{4} = 5$ ways to choose the subset $\{1,2,3,4,5\} \subset [5]$, and then 3 'double transpositions' on these four points.

3. $\phi(C) = \phi(A)^2\phi(B)^{-1} = \begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix}^2 \begin{bmatrix} -5 & -9 \\ 4 & 7 \end{bmatrix}^{-1} = \begin{bmatrix} 5 & 8 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 7 & 9 \\ -4 & -5 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -2 & -3 \end{bmatrix}$.

4. (a) We view G as the symmetry group of a regular n-gon, with g acting as a counter-clockwise rotation by an angle of $\frac{2\pi}{n}$, and h acting as a reflection in a line of symmetry passing through the center.

   (b) If we label the vertices of the regular n-gon as $1, 2, 3, \ldots, n$ in counter-clockwise order, then we may represent g by the n-cycle $(1, 2, 3, \ldots, n)$; and we may choose to represent h by $(1,n)(2,n{-}1)(3,n{-}2)\cdots(\frac{n-1}{2},\frac{n+1}{2})$ (if n is odd), or $(1,n)(2,n{-}1)(3,n{-}2)\cdots(\frac{n-2}{2},\frac{n+2}{2})$ (if n is even).

5. (a) T (b) T (c) F (d) F (e) T (f) F (g) F (h) F (i) F (j) T

Here are some remarks and partial explanations for answers in #5:

(a) If $(xy)^n = e$, then left-multiply by x^{-1} and right-multiply by x to obtain $(yx)^n = e$.

(b) Since $xy, y^{-1} \in \langle x, y\rangle$, we have $\langle xy, y^{-1}\rangle \leqslant \langle x, y\rangle$. Conversely, both of the elements $x = (xy)(y^{-1})$ and $y = (y^{-1})^{-1}$ are in $\langle xy, y^{-1}\rangle$, so $\langle x, y\rangle \leqslant \langle xy, y^{-1}\rangle$.

(c) A simple counterexample is $x = (1\,2)$ and $y = (1\,3)$ in S_3. There are also easy counterexamples when the group is abelian; e.g. $x = y = (1\,2)$ in S_2.

(d) Use the same counterexample as in (c).

(e) If x and y commute, then $\langle x, y\rangle = \{x^i y^j : i, j \in \mathbb{Z}\}$ and $(x^i y^j)(x^k y^\ell) = x^{i+k} y^{j+\ell} = (x^k y^\ell)(x^i y^j)$.

(f) Absolutely not. The symmetry group of a square is a set of transformations. It is not at all the same as the set of things that are being permuted or transformed. (*This is not that.*)

(g) The element $1 \in \mathbb{R}$ generates the additive subgroup $\mathbb{Z}$. This is far from being the entire real line (it does not include 1.47, for example). In fact, $\mathbb{R}$ is not cyclic at all. Every cyclic group is countable (either finite or countably infinite), whereas $\mathbb{R}$ is uncountable.

(h) As a counterexample, consider the multiplicative group consisting of all complex roots of unity (the set of all complex numbers z such that $z^n = 1$ for some positive integer n). Or take the additive group of infinite sequences $(a_0, a_1, a_2, a_3, \ldots)$ where $a_i \in \mathbb{F}_2$.

(i) In $\mathbb{F}_5$, $1{+}1{+}1{+}1{+}1 = 0$, whereas in $\mathbb{Z}$, $1{+}1{+}1{+}1{+}1 \neq 0$.

(j) Let G be a group of order at least 2. Then there exists a nonidentity element $g \in G$, so G has a cyclic subgroup $\langle g\rangle$ of order at least 2. Note that 1 and g are distinct elements of this sibgroup.