UNIVERSITY OF WYOMING

Math 4520—Fall 2025

Algebra III

Fields

Department of Mathematics

HW1

(Due 5:00pm Friday, September 26, 2025)

Instructions: Work by hand and calculator showing your work, and checking your answers whenever reasonably possible. All answers are exact: there are no approximate answers. If you have access to computer software, this may be used to *check* answers. Submit solutions through WyoCourses. See the syllabus and FAQ for general expectations regarding homework.

1. (*20 points*) Let $F = \mathbb{F}_{61} = \{0, 1, 2, \ldots, 60\}$ be the field of order 61 (the integers mod 61). Find the unique solution $(x, y) \in F^2$ of the linear system

$$\begin{cases} 31x + 5y = 8, \\ 44x + 51y = 51. \end{cases}$$

In #2 and #3, let $f(x) = x^3 - x - 1 \in \mathbb{Q}[x]$, and let $\theta \in \mathbb{C}$ be a root of f. (This information tells you that $f(\theta) = 0$, but you are *not* given the specific value of θ, not even whether or not θ is real, as this information is irrelevant.) You are given that $f(x)$ is irreducible in $\mathbb{Q}[x]$; and so as a consequence, you may take it as given that

$$F = \mathbb{Q}[\theta] = \{a + b\theta + c\theta^2 \,:\, a, b, c \in \mathbb{Q}\}$$

is a field. Every element of F has a *unique* expression of the form $a + b\theta + c\theta^2$ where $a, b, c \in \mathbb{Q}$. (This is simply to say that F is a three-dimensional vector space over $\mathbb{Q}$ with basis $\{1, \theta, \theta^2\}$.)

2. (*30 points*) Given $\alpha = \theta^2 - 2\theta \in F$ and $\beta = 2 + \theta \in F$, compute each of the following:

   (a) $\alpha + \beta$

   (b) $\alpha - \beta$

   (c) $\alpha\beta$

   (d) α/β

   Each of your answers should be expressed in simplified form as $a + b\theta + c\theta^2$ where $a, b, c \in \mathbb{Q}$ are reduced (i.e. simplified) fractions in $\mathbb{Q}$.

3. (*20 points*) Find the unique monic polynomial $m(x) \in \mathbb{Q}[x]$ of degree three having $\theta^2 + 1$ as a root.

Next, consider the 3×3 matrix $T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. The set of all 3×3 matrices with rational entries forms a ring $\mathbb{Q}^{3\times 3}$ (which is not a field). But the subset

$$E = \{aI + bT + cT^2 : a, b, c \in \mathbb{Q}\} \subset \mathbb{Q}^{3\times 3}$$

is a field (a fact which you may assume for now). Here we use the usual operations of scalar multiplication as well as matrix addition, subtraction and multiplication. The elements zero and one (in E) are just the zero matrix and the identity matrix of size 3×3. In #4, consider the two elements $A, B \in E$ given by $A = T^2 - 2T$ and $B = 2I + T$.

4. (*30 points*) Compute each of the following in E, simplifying your answers and expressing them in the standard form $aI + bT + cT^2 \in E$ where $a, b, c \in \mathbb{Q}$ are simplified:
   (a) $A + B$
   (b) $A - B$
   (c) AB
   (d) A/B