UNIVERSITY
OF WYOMING

Math 4520—Fall 2025
Algebra III
Fields

$F[\alpha]\cong F[t]/(f(t))$

Department of
Mathematics

Solutions to HW1

1. $$\left[\begin{array}{cc|c}31&5&8\\44&51&51\end{array}\right]\sim\left[\begin{array}{cc|c}1&10&16\\44&51&51\end{array}\right]\sim\left[\begin{array}{cc|c}1&10&16\\0&38&18\end{array}\right]\sim\left[\begin{array}{cc|c}1&10&16\\0&1&39\end{array}\right]\sim\left[\begin{array}{cc|c}1&10&16\\0&1&39\end{array}\right]\sim\left[\begin{array}{cc|c}1&0&53\\0&1&39\end{array}\right]$$

The unique solution is $(x,y)=(53,39)$. *Check*: $31{\cdot}53+5{\cdot}39=8$; $44{\cdot}53+51{\cdot}39=51$.

Rough work:
$$\begin{array}{ccc}61&31&\\\hline 1&0&61\\0&1&31\\1&-1&30\\-1&2&1\end{array}\qquad \frac{1}{31}=2\qquad\qquad \begin{array}{ccc}61&38&\\\hline 1&0&61\\0&1&38\\1&-1&23\\-1&2&15\\2&-3&8\\-3&5&7\\5&-8&1\end{array}\qquad \frac{1}{38}=-8=53$$

2. From $\theta^3=\theta+1$ we obtain $\theta^4=\theta^2+\theta$.

   (a) $\alpha+\beta\ =\ 2-\theta+\theta^2$.

   (b) $\alpha-\beta\ =\ -2-3\theta+\theta^2$.

   (c) $\alpha\beta\ =\ (-2\theta+\theta^2)(2+\theta)\ =\ -4\theta+\theta^3\ =\ -4\theta+(1+\theta)\ =\ 1-3\theta$.

   (d) $\frac{\alpha}{\beta}=a+b\theta+c\theta^2$ yields
   $$\begin{aligned}\alpha\ &=\ (a+b\theta+c\theta^2)(2+\theta)\\ &=\ 2a+(a{+}2b)\theta+(b{+}2c)\theta^2+c\theta^3\\ &=(2a{+}c)+(a{+}2b{+}c)\theta+(b{+}2c)\theta^2.\end{aligned}$$

   Solving this system of three linear equations gives the unique solution $(a,b,c)=(-\frac{4}{7},-\frac{9}{7},\frac{8}{7})$; thus $\frac{\alpha}{\beta}=-\frac{4}{7}-\frac{9}{7}\theta+\frac{8}{7}\theta^2$.

   Alternatively, $\beta=g(\theta)$ where $g(x)=2+x$. Now $\gcd(f(x),g(x))=1=-\frac{1}{7}f(x)+\frac{1}{7}(3-2x+x^2)g(x)$ by Euclid's Algorithm. Evaluate at θ to get $\beta^{-1}=\frac{1}{7}(3-2\theta+\theta^2)$. Multiply this by α to get $\alpha/\beta=\alpha\beta^{-1}=-\frac{4}{7}-\frac{9}{7}\theta+\frac{8}{7}\theta^2$.

3. We seek the minimal polynomial of $\gamma=\theta^2{+}1$, which we denote as $m(x)=x^3+ax^2+bx+c$. We simplify
   $$m(\gamma)\ =\ (a{+}b{+}c{+}2)+(a{+}5)\gamma+(3a{+}b{+}7)\gamma^2.$$
   To find the unique choice of $a,b,c\in\mathbb{Q}$ such that this expression vanishes, we simply solve the linear system of three equations in three unknowns to obtain $(a,b,c)=(-5,8,-5)$, so that $m(x)=x^3-5x^2+8x-5$.

Check: We may evaluate the expressions in #2,3 using any of the three roots of $f(x)$. For convenience, we try the real root $\theta\approx 1.324717957$, giving $\alpha\approx-0.894558248$, $\beta\approx 3.324717957$, $\gamma\approx 2.754877666$. This gives excellent numerical agreement for our answers in #2,3.

4. Observe that $T^3 = T + I$ where I is the 3×3 identity matrix; thus T is a root of $f(x)$. This means that $E = \mathbb{Q}[T] \cong \mathbb{Q}[\theta] = F$, the field in #2, 3. So by #2,

   (a) $A + B \;=\; 2I - T + T^2$.

   (b) $A - B \;=\; -2I - 3T + T^2$.

   (c) $AB \;=\; I - 3T$.

   (d) $A/B = AB^{-1} \;=\; -\frac{4}{7}I - \frac{9}{7}T + \frac{8}{7}T^2$.