UNIVERSITY OF WYOMING

Math 4520—Fall 2025

Algebra III

Fields

Department of Mathematics

HW2

Due Wednesday, October 29, 2025, by 5:00pm on WyoCourses

Instructions: Show your work, and *check* answers whenever possible. See the syllabus and FAQ for general expectations regarding homework. Total value of questions: 120 points.

1. (*20 points*) Factor each of the following polynomials into irreducible factors in $\mathbb{Z}[x]$.

   (a) $f(x) = x^4 + 2x^3 + 5x^2 + 4x + 3$

   (b) $g(x) = x^4 + 2x^3 + 3x^2 + 4x + 5$

   Recall: Neither of these polynomials is irreducible in $\mathbb{R}[x]$. For monic polynomials in $\mathbb{Z}[x]$ (such as these), the irreducible factors in $\mathbb{Z}[x]$ are the same as the irreducible factors in $\mathbb{Q}[x]$.

2. (*20 points*) Find the minimal polynomial (over $\mathbb{Q}$) for each of the following.

   (a) $\alpha = \sqrt{4 + \sqrt{3}}$

   (b) $\beta = \sqrt{37 + 20\sqrt{3}}$

In preparation for #3,4, let us evaluate the continued fraction $\alpha = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\cdots}}}}$ in closed form. Since $\alpha = 1 + \frac{1}{\alpha}$, we see that $\alpha^2 - \alpha - 1 = 0$ and so $\alpha = \frac{1\pm\sqrt{5}}{2}$. Evidently $\alpha > 0$, and so $\alpha = \frac{1+\sqrt{5}}{2}$, a quadratic irrational.

3. (*20 points*) Is $\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\text{etc.}}}}}$ rational, algebraic irrational or transcendental? Justify your answer.

4. (*20 points*) Let $x = \sqrt{5 + \sqrt{5 - \sqrt{5 + \sqrt{5 - \sqrt{5 + \sqrt{5 - \cdots}}}}}}$. Show that x is algebraic, and find its minimal polynomial over $\mathbb{Q}$.

5. (*20 points*) Let $\alpha = 2\cos\frac{2\pi}{5} = 2\cos 72°$. Imitating computations done in class, show that $\alpha = \frac{-1+\sqrt{5}}{2}$. (Note that because this is a quadratic irrational, a regular pentagon is constructible using straightedge and compass.)

6. (*20 points*) The field of order 49 can be viewed as $E = \mathbb{F}_{49} = F[i] = \{a+bi : a, b \in F\}$ where $F = \mathbb{F}_7$ and $i = \sqrt{-1}$. Factor each of the following polynomials into linear factors in $E[x]$; and in each case state the two roots in E.

   (a) $x^2 + 3x + 3$

   (b) $x^2 + 3x + 5$