



## Solutions to HW2

1. (a)  $f(x) = x^4 + 2x^3 + 5x^2 + 4x + 3 = (x^2 + x + 1)(x^2 + x + 3)$  where the quadratic factors are irreducible in  $\mathbb{Z}[x]$  (also in  $\mathbb{Q}[x]$  and in  $\mathbb{R}[x]$ ) since they have negative discriminants  $-3$  and  $-11$  respectively.
- (b)  $g(x) = x^4 + 2x^3 + 3x^2 + 4x + 5$  is irreducible in  $\mathbb{Z}[x]$  (and in  $\mathbb{Q}[x]$ ). If  $g(x)$  had a linear factor in  $\mathbb{Z}[x]$ , then  $g(x) = (x - r)(x^3 + sx^2 + tx + u)$  where  $r, s, t, u \in \mathbb{Z}$ ; but then  $r \in \{\pm 1, \pm 5\}$ . This is impossible since  $g(r) = 15, 3, 975, 435$  for  $r = 1, -1, 5, -5$  respectively. On the other hand, if  $g(x)$  factors into quadratic factors in  $\mathbb{Z}[x]$ , then  $g(x) = (x^2 + ax \pm 1)(x^2 + bx \pm 5)$  for some  $a, b \in \mathbb{Z}$ . This yields  $a + b = 2$  and  $5a + b = \pm 4$ , whence  $4a = \pm 4 - 2 \in \{2, -6\}$ , which is impossible. So in this case the irreducible factorization has just one factor  $g(x)$ .
2. (a) Since  $\alpha^2 = 4 + \sqrt{3}$ , we have  $\alpha^4 - 8\alpha^2 + 16 = (\alpha^2 - 4)^2 = 3$  and so  $\alpha^4 - 8\alpha^2 + 13 = 0$ . In order to check that  $m(x) = x^4 - 8x^2 + 13$  is the minimal polynomial of  $\alpha$ , it remains to be checked that  $m(x)$  is irreducible in  $\mathbb{Z}[x]$  (and so also in  $\mathbb{Q}[x]$ ). If  $\alpha$  has any factorization of the form  $m(x) = (x - r)(x^3 + sx^2 + tx + u)$  where  $r, s, t, u \in \mathbb{Z}$ , then  $ru = 13$  and  $r \in \{\pm 1, \pm 13\}$  is a root of  $m(x)$ . But this is not possible since  $m(\pm 1) = 6$  and  $m(\pm 13) = 27222$ .

Any other factorization of  $m(x)$  has two quadratic factors, and this gives  $m(x) = (x^2 + ax + b)(x^2 - ax + c)$  where  $a, b, c \in \mathbb{Z}$ . (The coefficients  $a, -a$  are required in order that  $m(x)$  have no  $x^3$  term.) But then  $a(b - c) = 0$  and  $bc = 13$ . From the last relation, one of  $\{b, c\}$  is  $\pm 1$  and the other is  $\pm 13$ . In particular,  $b \neq c$ ; so the previous relation gives  $a = 0$ . But then  $m(x) = (x^2 + b)(x^2 + c)$  and  $b + c = -8$ . However, this is inconsistent with  $\{b, c\} = \{\pm 1, \pm 13\}$ . This is a final contradiction.

We conclude that  $m(x) = x^4 - 8x^2 + 13$  is the minimal polynomial of  $\alpha$  over  $\mathbb{Q}$ .

- (b) Proceeding as in (a),  $\beta^2 = 37 + 20\sqrt{3}$  gives  $\beta^4 - 74\beta^2 + 1369 = (\beta^2 - 37)^2 = (20\sqrt{3})^2 = 1200$  and so  $\beta$  is a root of  $g(x) = x^4 - 74x^2 + 169$ . But this time,  $g(x)$  is reducible. We may discover this by the method of (a), obtaining a factorization  $g(x) = (x^2 + 10x + 13)(x^2 - 10x + 13)$ . Alternatively, this may be discovered by completing the square:

$$\begin{aligned}
g(x) &= x^4 - 74x^2 + 169 = (x^2 + 13)^2 - 26x^2 - 74x^2 = (x^2 + 13)^2 - 100x^2 \\
&= (x^2 + 13 + 10x)(x^2 + 13 - 10x).
\end{aligned}$$

Now it is easy to check that  $\beta$  is a root of the second factor  $x^2 - 10x + 13$  and that this is irreducible, so this is in fact the minimal polynomial of  $\beta$ .

3. The indicated value satisfies  $x = \sqrt{2}^x = 2^{x/2}$ , so  $\ln x = \ln(2^{x/2}) = \frac{x}{2} \ln 2$  and  $\ln \frac{x}{2} = \frac{x}{2}$ , i.e.  $g(x) = g(2)$  where  $g(x) = \frac{\ln x}{x}$ . By inspection, this has two solutions  $x \in \{2, 4\}$ . Using first-semester calculus, we can see that these are the only solutions (since  $g$  is increasing on  $(0, e)$  and decreasing on  $(e, \infty)$ ). The correct (or intended) answer is  $x = 2$ . By the explanation given in class,  $x$  should be understood as the limit of the sequence

$$1, \sqrt{2}, \sqrt{2}^{\sqrt{2}}, \sqrt{2}^{\sqrt{2}^{\sqrt{2}}}, \dots$$

It is not hard to check (again using calculus) that this is an increasing sequence with terms less than 2, so the limit cannot be 4. Since it is a bounded increasing sequence, its limit exists and this limit must equal 2. But since this is an algebra class and not a calculus (or analysis) class, I feel compelled to give full credit for either of the answers 2 or 4. Either way, this value is of course **rational**.

Here is a more complete explanation. Define  $f(x) = 2^{x/2} = \sqrt{2}^x$ . So the required value is the limit of the sequence  $1, f(1), f(f(1)), f(f(f(1))), \dots$ . Whenever  $0 < x < 2$ , we have  $0 < x < f(x) < 2$ . (Use  $f(x) = 2^{x/2} < 2^1 = 2$ . Also by the first derivative test, the function  $g(x) = \frac{\ln x}{x}$  is increasing on the interval  $(0, e]$ , so  $\frac{\ln x}{x} < \frac{\ln 2}{2}$  and  $x = e^{\ln x} < e^{(x \ln 2)/2} = f(x)$ .) Our sequence of approximations is  $1, f(1), f(f(1)), f(f(f(1))), \dots$  which is an increasing sequence of real numbers less than 2. So it converges to a real number  $\alpha \leq 2$ . This number satisfies  $\alpha = f(\alpha)$ , so  $\ln \alpha = \ln f(\alpha) = \frac{\alpha \ln 2}{2}$ , i.e.  $g(\alpha) = g(2)$ . Again using the fact that  $g(x)$  is increasing on the interval  $(0, e]$ , this forces  $\alpha = 2$ . Of course this value is rational.

Similarly in #4, I will not require the full explanation of why the limit exists; full credit was given for algebraically solving the formula for  $\alpha$ .

4. I regret writing ‘ $x$ ’ instead of ‘ $\alpha$ ’ for the constant in the question. Although the choice of letters doesn’t truly matter in any absolute sense, this confused some students because it deviates from the convention I have been following in the course (writing  $\alpha, \beta, \gamma, \dots$  for constants in the field; and  $x, y, z, t, \dots$  for indeterminates). I hope you will understand if I switch back now and write  $\alpha$  instead of  $x$ .

The required value  $\alpha$  satisfies  $\alpha = \sqrt{5 + \sqrt{5 - \alpha}}$ , so  $\alpha^2 = 5 + \sqrt{5 - \alpha}$ ,  $\alpha^2 - 5 = \sqrt{5 - \alpha}$  and  $\alpha^4 - 10\alpha^2 + 25 = 5 - \alpha$ . So  $\alpha$  is a root of

$$t^4 - 10t^2 + t + 20 = (t^2 + t - 5)(t^2 - t - 4).$$

Thus  $\alpha \in \{\frac{1}{2}(-1 \pm \sqrt{21}), \frac{1}{2}(1 \pm \sqrt{17})\}$ . However, it is evident from the original expression for  $\alpha$  that we must have  $\alpha > \sqrt{5}$ . Only one of the four roots satisfies this requirement, giving  $\alpha = \frac{1}{2}(1 + \sqrt{17})$ . This value is algebraic of degree two (a quadratic irrational) with minimal polynomial  $t^2 - t - 4$  over  $\mathbb{Q}$ . (*Remarks:* I have seen this value of  $\alpha$  arise naturally in a combinatorial counting problem....)

5. We start with a primitive complex fifth root of unity  $\zeta = e^{2\pi i/5}$ , so that  $\zeta^5 = 1$  and  $\zeta^4 + \zeta^3 + \zeta^2 + \zeta + 1 = 0$ . Now  $\alpha = \zeta + \zeta^{-1}$  and  $\alpha^2 = \zeta^2 + 1 + \zeta^{-2}$ , so

$$\alpha^2 + \alpha - 1 = \zeta^2 + \zeta + 1 + \zeta^{-1} + \zeta^{-2} = 0.$$

Thus  $\alpha$  is a root of  $m(x) = x^2 + x - 1 \in \mathbb{Q}[x]$  which is irreducible over  $\mathbb{Q}$ , having roots  $\frac{1}{2}(-1 \pm \sqrt{5})$ . Since  $\alpha = 2 \cos \frac{2\pi}{5} > 0$ , we must have  $\alpha = \frac{1}{2}(-1 + \sqrt{5}) \approx 0.618$ .

6. (a) The roots are  $\frac{1}{2}(-3 \pm \sqrt{9 - 12}) = \frac{1}{2}(-3 \pm \sqrt{4}) = \frac{1}{2}(-3 \pm 2) = 1, 3$ . Thus

$$x^2 + 3x + 3 = (x + 4)(x + 6).$$

Alternatively, the polynomial  $x^2 + 3x + 3$  has values 3, 0, 6, 0, 3, 1, 1 at the points 0, 1, 2, 3, 4, 5, 6  $\in F$ .

- (b) The roots are  $\frac{1}{2}(-3 \pm \sqrt{9 - 20}) = \frac{1}{2}(-3 \pm \sqrt{-4}) = \frac{1}{2}(-3 \pm 2i) = 2 \pm i$ . Thus

$$x^2 + 3x + 5 = (x - 2 - i)(x - 2 + i).$$