University of Wyoming

Math 4520—Fall 2025

Algebra III

Fields

Department of Mathematics

HW3

Due 5:00pm, Wednesday, December 3, 2025, on WyoCourses

Instructions: Show your work, and *check* answers whenever possible. Submit solutions through WyoCourses. See the syllabus and FAQ for general expectations regarding homework. Total value of questions: 65 points.

In #1, take $F = \mathbb{F}_2 = \{0,1\}$ and $E = \mathbb{F}_8$, an extension of degree $[E:F] = 3$ with basis $\{1, \theta, \theta^2\}$ where $\theta^3 = \theta + 1$. The minimial polynomial of θ over F is $m(x) = x^3 + x + 1$. The polynomial $m(x)$ has three roots in E, namely θ, θ^2, θ^4, from which we obtain the factorization $m(x) = (x-\theta)(x-\theta^2)(x-\theta^4)$. We have

$$E = \mathbb{F}_8 = \{a + b\theta + c\theta^2 \,:\, a,b,c \in \mathbb{F}_2\} = \{0, 1, \theta, \theta^2, \theta^3, \theta^4, \theta^5, \theta^6\}.$$

Here we tabulate the powers of θ:

k	0	1	2	3	4	5	6
θ^k	1	θ	θ^2	$1+\theta$	$\theta+\theta^2$	$1+\theta+\theta^2$	$1+\theta^2$

Also note that $\operatorname{Aut} E = \{\iota, \sigma, \sigma^2\}$ is cyclic of order 3 with $\sigma^3 = \iota$, where $\sigma(x) = x^2$ for all $x \in E$.

1. (*20 points*) Consider the polynomial $h(x) = x^8 - x$,
   (a) Factor $h(x)$ into irreducible factors in $F[x]$.
   (b) Factor $h(x)$ into irreducible factors in $E[x]$.

2. (*45 points*) Let $f(x) = x^4 - 4x^2 + 2$.
   (a) Show that $f(x)$ is irreducible in $\mathbb{Z}[x]$ (and hence in $\mathbb{Q}[x]$).
   (b) Show that $f(x)$ has four roots $\pm\alpha$, $\pm\beta$ where $\alpha = \sqrt{2+\sqrt{2}}$ and $\beta = \sqrt{2-\sqrt{2}}$.
   (c) Let $E = \mathbb{Q}[\alpha]$. Show that $\sqrt{2} \in E$, by explicitly writing $\sqrt{2}$ as a polynomial in α with rational coefficients.
   (d) Show that $\beta \in E$, by explicitly writing β as a polynomial in α with rational coefficients.
   (e) Using (d), explain why $\mathbb{Q}[\beta] = E$.
   (f) According to the facts above, $[E:\mathbb{Q}] = 4$. The field E has an automorphism σ satisfying $\sigma(\alpha) = \beta$. (You may assume this.) Determine $\sigma(\sqrt{2})$ in simplified form.
   (g) Determine $\sigma(\beta)$ in simplified form.
   (h) The group $G = \operatorname{Aut} E$ has order 4. Is this a Klein four-group, or a cyclic group of order 4? Explain.
   (i) How many subfields does E have? What are they?