Sample Test

October, 2025

This sample test is intended to resemble the upcoming Test (Monday, November 10) in approximate length, difficulty, and style, although clearly the content may differ. The actual content will consist of material covered in class and on our handouts.

Instructions. The only aids allowed are a hand-held calculator and one 'cheat sheet', i.e. an $8.5'' \times 11''$ sheet with information written on one side in your own handwriting. No cell phones are permitted (in particular, a cell phone may not be used as a calculator). Answer as clearly and precisely as possible. *Clarity is required for full credit!* Total value of questions: 100 points (plus 15 bonus points).

1. *(20 points)* Let $\zeta = e^{i\pi/3}$. Note that the complex primitive sixth roots of unity are ζ and ζ^5.
   (a) Determine $m(x)$, the minimal polynomial of ζ over $\mathbb{Q}$.
   (b) Let $E = \mathbb{Q}[\zeta]$. Determine the degree $[E : \mathbb{Q}]$ of this extension, and find an explicit basis for E over $\mathbb{Q}$.
   (c) Show that $\sqrt{-3} \in E$.
   (d) How many subfields does E have? What are they?

2. *(20 points)* Using the fact that π is transcendental, show that $\sqrt{\pi}$ and π^2 are transcendental.

3. *(20 points)* Let $\alpha \in \mathbb{C}$ be algebraic with minimal polynomial $f(x) = x^3+3x^2+x+1 \in \mathbb{Q}[x]$. (You may use the fact that $f(x)$ is irreducible in $\mathbb{Q}[x]$.) Determine the minimal polynomial of $2\alpha+1$ (in simplified form).

4. *(10 points)* If α is as in #3, how many subfields does $\mathbb{Q}[\alpha]$ have? What are they? Justify your answer.

5. *(15 points)* If α is as in #3, then $\alpha^{-1} \in \mathbb{Q}[\alpha] = \{a + b\alpha + c\alpha^2 \ : \ a, b, c \in \mathbb{Q}\}$. Find explicit values of $a, b, c \in \mathbb{Q}$ such that $\alpha^{-1} = a + b\alpha + c\alpha^2$.

6. *(30 points)* Answer TRUE or FALSE to each of the following statements.

(a) Every subfield of $\mathbb{R}$ is infinite. ______*(True/False)*

(b) The only extension field $E \supseteq \mathbb{C}$ is $E = \mathbb{C}$ itself. ______*(True/False)*

(c) If F and F' are subfields of a field E, then $F \cap F'$ is also a subfield of E. ______*(True/False)*

(d) If F is a field of order 16, then F has subfields of order 2, 4, 8, and 16.

(e) If $F, F' \subset \mathbb{C}$ are subfields with both $[F : \mathbb{Q}]$ and $[F' : \mathbb{Q}]$ finite, then there exists a finite extension field $E \supseteq \mathbb{Q}$ containing both F and F'. ______*(True/False)*

(f) If $\alpha \in \mathbb{C}$ satisfies $\alpha^2 - \alpha - 3 = 0$, then $\mathbb{Q}[\alpha] = \mathbb{Q}[\sqrt{13}]$. ______*(True/False)*

(g) If $\zeta \in \mathbb{C}$ is a primitive n-th root of unity, then $[\mathbb{Q}[\zeta] : \mathbb{Q}] = n$. ______*(True/False)*

(h) If S is the set of all invertible 2×2 matrices with rational entries, then $S \cup \left\{ \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right] \right\}$ is a field. ______*(True/False)*

(i) For every positive integer n, there is at least one extension $E \supseteq \mathbb{Q}$ of degree $[E : \mathbb{Q}] = n$. ______*(True/False)*

(j) The value $\alpha = \sqrt{2} + \sqrt{3}$ is algebraic of degree 4 over $\mathbb{Q}$, i.e. $[\mathbb{Q}[\alpha] : \mathbb{Q}] = 4$. ______*(True/False)*