University of Wyoming

Math 4520—Fall 2025

Algebra III

Fields

Department of Mathematics

Optional Practice Problems

December, 2025

These problems are intended as practice leading up to the final exam (8:00–10:00 am on Wednesday, December 10, 2025 in our usual lecture room, AG 4021). The actual final exam will have somewhat easier questions than this practice exam, and about half as many questions. Its content will, however, consist of material covered in class this semester, and all related handouts. Somewhat greater weight will be placed on the later material (covered after the Test).

Instructions. The only aids allowed are a hand-held calculator and one 'cheat sheet', i.e. an 8.5″×11″ sheet with information written on one side in your own handwriting. No cell phones are permitted (in particular, a cell phone may not be used as a calculator). Answer as clearly and precisely as possible. *Clarity is required for full credit!* Time permitted: 120 minutes. Total value of questions: 100 points. (Just as in our previous Test, there will be bonus points available.)

1. Consider the extension $E \supset F$ where $E = \mathbb{F}_{25}$ and $F = \mathbb{F}_5$, and let $\alpha \in E$ be a root of $x^2 + 2x + 3 \in F[x]$. Find the minimal polynomial of $\beta = 2\alpha + 1$ over F. Simplify your answer.

2. Determine the minimal polynomial of $\alpha = 2^{1/3} + 2^{2/3}$ over $\mathbb{Q}$.

3. Let $\alpha = \sqrt{5} + \sqrt{22 + 2\sqrt{5}}$.
   (a) Find the minimal polynomial of α over $\mathbb{Q}$.
   (b) Prove that $\sqrt{11 + 2\sqrt{29}} + \sqrt{16 - 2\sqrt{29} + 2\sqrt{55 - 10\sqrt{29}}} = \alpha$.

4. Let $\alpha, \beta, \gamma \in \mathbb{C}$ be the three roots of $m(x) = x^3 - 7x^2 + 5x + 3$. Compute each of the following. Each of your answers should be expressed simply as an integer.
   (a) $\alpha + \beta + \gamma$
   (b) $\alpha\beta\gamma$
   (c) $\alpha^2 + \beta^2 + \gamma^2$

5. Give an explicit construction of a field of order 16.

6. Let $F = \mathbb{F}_p$ where p is prime. How many *irreducible* polynomials $f(x) \in F[x]$ of degree two are there? And how many of these are monic?

7. Does the field $\mathbb{Q}[\sqrt{-2}\,]$ contain $i = \sqrt{-1}$? Justify your answer.

8. Let $\alpha = 2^{1/3}$. Show that α cannot be expressed as a rational linear combination of square roots of rational numbers; i.e. for any rational numbers a_i, b_i $(i = 1, 2, \ldots, n)$, the value of $a_1\sqrt{b_1} + a_2\sqrt{b_2} + \cdots + a_n\sqrt{b_n}$ cannot equal α.

9. Are the fields $\mathbb{Q}[\sqrt{3}]$ and $\mathbb{Q}[\sqrt{5}]$ isomorphic? Justify your answer.

10. Let ζ be a complex root of $f(x) = x^4 + 1 \in \mathbb{Q}[x]$. Show that $f(x)$ is irreducible in $\mathbb{Q}[x]$, and that $f(x) = (x - \zeta)(x - \zeta^3)(x - \zeta^5)(x - \zeta^7)$. Find all automorphisms of the field $E = \mathbb{Q}[\zeta]$. Denote by G the group of all automorphisms of $E = \mathbb{Q}[\zeta]$. Show that $|G| = 4$. Is the group G cyclic, or is it a Klein four-group? Explain.

11. Answer TRUE or FALSE to each of the following statements.

    (a) If σ and τ are automorphisms of a field F, then necessarily $\sigma\tau = \tau\sigma$.
    _____*(True/False)*

    (b) If $K \supseteq \mathbb{Q}$ is a finite extension field, then every automorphism of K is continuous.
    _____*(True/False)*

    (c) For every finite field F of order q, the nonzero elements of F form a multiplicative group of order $q - 1$ which is cyclic. _____*(True/False)*

    (d) There is an automorphism of $\mathbb{R}$ mapping $\sqrt{2} \mapsto -\sqrt{2}$. _____*(True/False)*

    (e) There exist subfields $K, L \subseteq \mathbb{R}$ such that $K \neq L$ but $K \cong L$. _____*(True/False)*

    (f) There exists a proper extension field $E \supset \mathbb{C}$. (Recall that 'proper' means $E \neq \mathbb{C}$.)
    _____*(True/False)*

    (g) Let $E \supseteq \mathbb{Q}$ be an extension field. Then every automorphism of E is a linear transformation of the vector space E over the prime field $\mathbb{Q}$. _____*(True/False)*

    (h) If $a \in \mathbb{C}$ is transcendental over $\mathbb{Q}$, then necessarily so are $a + 1$ and a^2.
    _____*(True/False)*

    (i) A regular 60-gon can be constructed using only a straightedge and compass.
    _____*(True/False)*

    (j) Let $E \supset \mathbb{Q}$ be a cubic field extension, so that $[E : \mathbb{Q}] = 3$. Suppose that E has three distinct automorphisms ι, σ, σ^2. Then $a + \sigma(a) + \sigma^2(a) \in \mathbb{Q}$ for every $a \in E$.
    _____*(True/False)*