Ad Hoc Lectures

$$a = 1.3564547206166 = [1,2,1,4,7,5] = [1,2,1,4,7,4,1] = 1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{7+\cfrac{1}{4+1}}}}}$$

$$= 1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{7+\frac{1}{5}}}}} = 1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{4+\frac{5}{36}}}} = 1+\cfrac{1}{2+\cfrac{1}{1+\frac{36}{149}}} = 1+\cfrac{1}{2+\frac{149}{185}} = 1+\frac{185}{519} = \frac{704}{519}$$

Check!

$$704 = 1\times 519 + 185$$
$$519 = 2\times 185 + 149$$
$$185 = 1\times 149 + 36$$
$$149 = 4\times 36 + 5$$
$$36 = 7\times 5 + \boxed{1}$$
$$5 = 5\times 1 + 0$$

Continued fraction expansion

$$[a_1, a_2, a_3, a_4, \cdots, a_k] = a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{1}{a_4+\cdots+\frac{1}{a_k}}}}$$

Infinite continued fraction expansion

$$[a_1, a_2, a_3, \cdots] = \underbrace{\lim_{k\to\infty}[a_1, \cdots, a_k]}$$

limit of rational numbers converging to the real number shown

Every real number has a continued fraction expansion which is (essentially) unique. This expansion is finite whenever we started with a rational number; infinite expansion if the number was irrational.

$\pi = 3.14159265358979323\ldots = [3, 7, 15, 1, 292, 1, 1, 1, 2, \ldots]$
is the limit of the sequence of rational numbers (known as the continued fraction convergents)

$$[3] = 3$$

$$[3, 7] = 3 + \frac{1}{7} = \frac{22}{7}$$

$$[3, 7, 15] = 3 + \cfrac{1}{7 + \cfrac{1}{15}} = \frac{333}{106}$$

$$[3, 7, 15, 1] = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1}}} = \frac{355}{113}$$

$$a = 2.2602428874665\cdots = [2,3,1,5,2,1,5,2,1,5,2,\cdots] = [2,3,\overline{1,5,2}]$$

$$b = [\overline{1,5,2}] = [1,5,2,1,5,2,\cdots] = 1+\cfrac{1}{5+\cfrac{1}{2+\cfrac{1}{b}}} = 1+\cfrac{1}{5+\cfrac{b}{2b+1}} = 1+\frac{2b+1}{11b+5}$$

$$b-1 = \frac{2b+1}{11b+5}$$

$$11b^2-6b-5 = 2b+1$$

$$11b^2-8b-6=0$$

$$b = \frac{8\pm\sqrt{64+264}}{22} = \frac{8\pm\sqrt{328}}{22} = \frac{4\pm\sqrt{82}}{11} = \frac{4+\sqrt{82}}{11}$$

$$a = 2+\cfrac{1}{3+\cfrac{1}{\underbrace{1+\cfrac{1}{5+\cfrac{1}{2+\cfrac{1}{\cdots}}}}_{b}}} = 2+\cfrac{1}{3+\cfrac{1}{b}} = 2+\frac{b}{3b+1} = \frac{7b+2}{3b+1} = \frac{7\,\frac{4+\sqrt{82}}{11}+2}{3\,\frac{4+\sqrt{82}}{11}+1}\cdot\frac{11}{11}$$

$$a = \frac{28+7\sqrt{82}+22}{12+3\sqrt{82}+11} = \frac{50+7\sqrt{82}}{23+3\sqrt{82}}\cdot\frac{23-3\sqrt{82}}{23-3\sqrt{82}} = \frac{-572+11\sqrt{82}}{-209} = \frac{572-11\sqrt{82}}{209} = \frac{52-\sqrt{82}}{19}$$ (check!)

$$\sqrt{2} = [1,2,2,2,2,2,\cdots]$$

Pell's Equation $x^2 - dy^2 = \pm 1$ (d positive integer) has infinitely many solutions $(x,y) \in \mathbb{Z}^2$ (if d is squarefree)

eg. $x^2 - 29y^2 = 1$. $(x,y) = \underbrace{(\pm 1, 0)}_{\text{trivial solutions}}, (\pm 9801, \pm 1820), \ldots$

$(9801, 1820)$ is the fundamental solution

Solutions
Found using continued fractions

$$\sqrt{29} = [5, 2, 1, 1, 2, 10, 2, 1, 1, 2, 10, 2, 1, 1, 2, 10, \cdots] = [5, \overline{2, 1, 1, 2, 10}]$$

Convergents:

$\frac{5}{1} \leftarrow 5^2 - 29 \cdot 1^2 = -4$

$\frac{11}{2} \leftarrow 11^2 - 29 \cdot 2^2 = 5$

$\frac{16}{3} \leftarrow 16^2 - 29 \cdot 3^2 = -5$

$\frac{27}{5} \leftarrow \text{-----} = 4$

$\frac{70}{13} \leftarrow \text{----} = -1$

$\frac{727}{135} \leftarrow \text{-----} = 4$

$\frac{1524}{283} \leftarrow \text{---} -5$

$\frac{2251}{418} \leftarrow \cdots\ 5$

$\frac{3775}{70} \leftarrow \cdots = -4$

$\frac{9801}{1820} \leftarrow 9801^2 - 29 \cdot 1820^2 = 1$

To solve $x^2 - dy^2 = 1$

note: $\left(\frac{x}{y}\right)^2 - d = \frac{1}{d^2} \approx 0$

$\left(\frac{x}{y}\right)^2 \approx d$

$\frac{x}{y} \approx \sqrt{d}$

$a = 1.2181818\cdots = 1.2\overline{18}$

$1000a = 1218.181818\cdots$

$10a = 12.181818\cdots$

$990a = 1206$

$a = \frac{1206}{990} = \frac{67}{55}$ (check!)

$a = 1.3564547206166$ Find $a \in \mathbb{Q}$.

$$= 1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{7+\frac{1}{5}}}}} = 1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{4+\frac{5}{36}}}} = 1+\cfrac{1}{2+\cfrac{1}{1+\frac{36}{149}}} = 1+\cfrac{1}{2+\frac{149}{185}} = 1+\frac{185}{519} = \frac{704}{519} \text{ (check!)}$$

$$1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{7+\cfrac{1}{4+\frac{1}{1}}}}}} = [1,2,1,4,7,4,1]$$

$704 = 1\times 519 + 185$

$519 = 2\times 185 + 149$

$185 = 1\times 149 + 36$

$149 = 4\times 36 + 5$

$36 = 7\times 5 + \boxed{1} = \gcd(704, 519)$

$5 = 5\times 1 + 0$

A (finite) continued fraction expansion has the form

$$[a_1, a_2, \cdots, a_k] = a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{1}{a_4+\cdots+\frac{1}{a_k}}}}$$

An (infinite) continued fraction expansion has the form

$$[a_1, a_2, a_3, a_4, \cdots] = a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{1}{a_4+\cfrac{1}{a_5+\cdots}}}} = \lim_{k\to\infty} [a_1, a_2, \cdots, a_k]$$

These rational approximations converge to $[a_1, a_2, a_3, a_4, \cdots]$ (the <u>continued fraction convergents</u>)

Eg. $\pi = [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, \dots]$

Continued fraction convergents to π:

$$[3] = 3$$

$$[3,7] = 3 + \frac{1}{7} = \frac{22}{7}$$

$$[3,7,15] = 3 + \cfrac{1}{7 + \cfrac{1}{15}} = \frac{333}{106}$$

$$[3,7,15,1] = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1}}} = \frac{355}{113}$$

$$\vdots$$

$$a = 2.26024288746645\ldots = 2 + \cfrac{1}{3 + \cfrac{1}{b}} = 2 + \frac{b}{3b+1} = \frac{7b+2}{3b+1} = \frac{7\,\frac{4+\sqrt{82}}{11} + 2}{3\,\frac{4+\sqrt{82}}{11} + 1} \cdot \frac{11}{11} = \frac{28 + 7\sqrt{82} + 22}{12 + 3\sqrt{82} + 11}$$

$$= [2, 3, 1, 5, 2, 1, 5, 2, 1, 5, 2, \cdots]$$

$$= [2, 3, \overline{1, 5, 2}]$$

$$= \frac{50 + 7\sqrt{82}}{23 + 3\sqrt{82}} \cdot \frac{23 - 3\sqrt{82}}{23 - 3\sqrt{82}} = \frac{-572 + 11\sqrt{82}}{-209} = \frac{572 - 11\sqrt{82}}{209} = \frac{52 - \sqrt{82}}{19}$$

(check!)

$$b = [\overline{1, 5, 2}] = [1, 5, 2, 1, 5, 2, \cdots] = 1 + \cfrac{1}{5 + \cfrac{1}{2 + \cfrac{1}{b}}} = 1 + \cfrac{1}{5 + \cfrac{b}{2b+1}} = 1 + \frac{2b+1}{11b+5}$$

$$b - 1 = \frac{2b+1}{11b+5}$$

$$(b-1)(11b+5) = 2b+1$$

$$11b^2 - 6b - 5 = 2b + 1$$

$$11b^2 - 8b - 6 = 0$$

$$b = \frac{8 \pm \sqrt{64 + 264}}{22} = \frac{8 \pm \sqrt{328}}{22} = \frac{4 \pm \sqrt{82}}{11} = \frac{4 + \sqrt{82}}{11}$$

A quadratic irrational is a root of an irreducible quadratic over $\mathbb{Q}$ i.e. $ax^2 + bx + c$
$a, b, c \in \mathbb{Q}$, $b^2 - 4ac \notin$ (perfect) square

Quadratic irrationals have repeating continued fraction expansion.

Solution of Pell's equation $x^2 - 29y^2 = 1$

Infinitely many solutions $(x,y) = (\pm 1, 0)$, $(\pm 9801, \pm 1820)$,

(trivial solutions) fundamental solution $(9801, 1820)$

Can be found using continued fractions:

$$\sqrt{29} = [5, \overline{2, 1, 1, 2, 10}]$$

Convergents:

$[5] = \frac{5}{1} \longleftarrow 5^2 - 29 \cdot 1^2 = -4$

$[5,2] = 5 + \frac{1}{2} = \frac{11}{2} \longleftarrow 11^2 - 29 \cdot 2^2 = 5$

$[5,2,1] = \frac{16}{3} \longleftarrow 16^2 - 29 \cdot 3^2 = -5$

$\frac{27}{5} \longleftarrow$ 4

$\frac{70}{13} \longleftarrow$ -1

$\frac{727}{135} \longleftarrow$ 4

$\frac{1524}{283} \longleftarrow$ -5

$\frac{2251}{418} \longleftarrow$ 5

$\frac{3775}{701} \longleftarrow$ -4

$\frac{9801}{1820} \longleftarrow 9801^2 - 29 \cdot 1820^2 = 1$

$\vdots$

$$9801^2 - 29 \cdot 1820^2 = 1$$

$$\left(\frac{9801}{1820}\right)^2 - 29 = \frac{1}{1820^2}$$

$$\Rightarrow \frac{9801}{1820} \approx \sqrt{29}$$