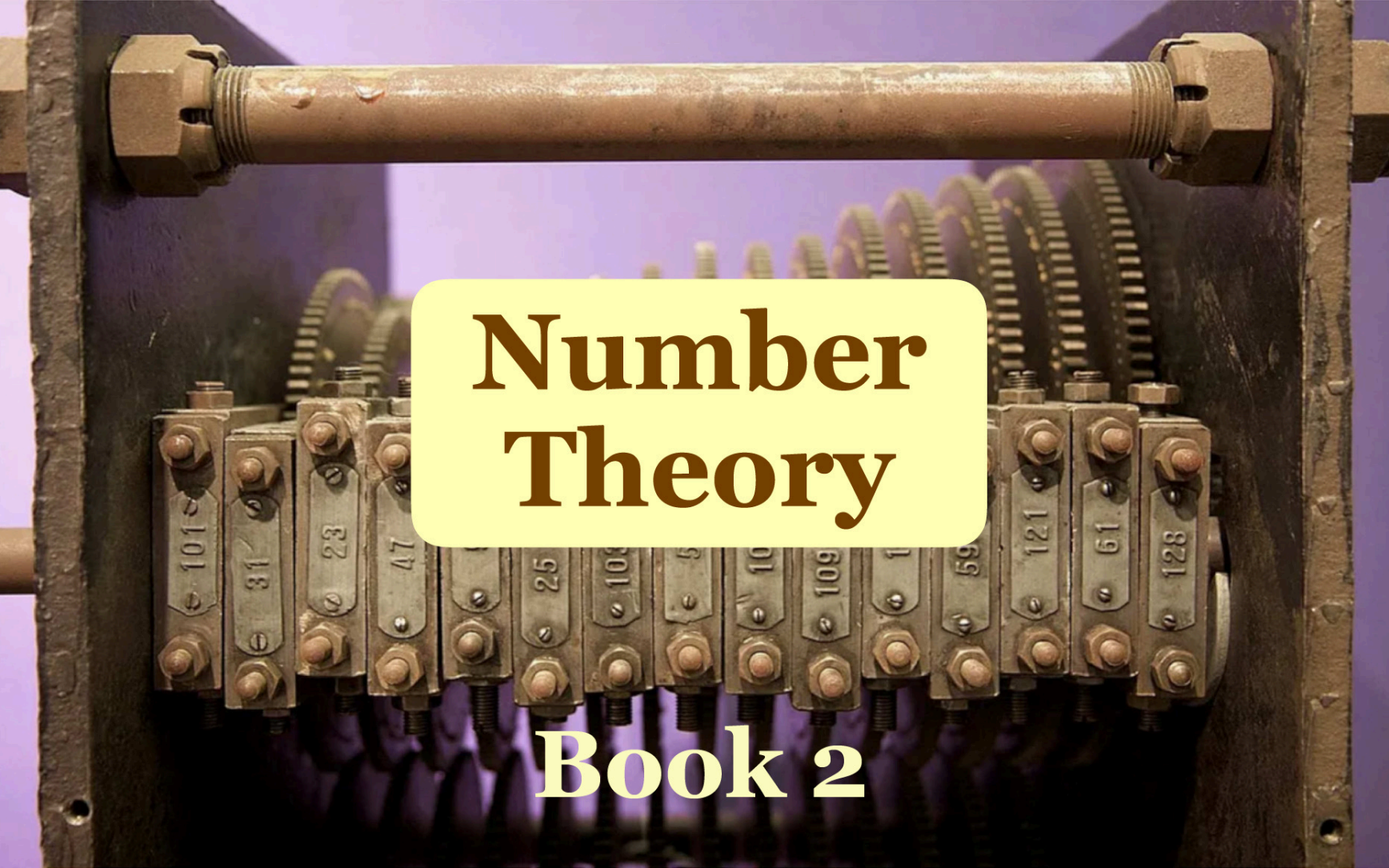
Number
Theory
Book 2

"Trick" (or technique) for identifying which small integers have the form x^2+y^2 or x^2+3y^2 or ...

Define $\theta(t) = \sum_{n=-\infty}^{\infty} t^{n^2} = 1 + 2t + 2t^4 + 2t^9 + 2t^{16} + 2t^{25} + \cdots = \sum_{n\in\mathbb{Z}} t^{n^2}$

$\theta(t)^2 = (1+2t+2t^4+2t^9+2t^{16}+2t^{25}+\cdots)(1+2t+2t^4+2t^9+2t^{16}+2t^{25}+\cdots) = 1 + 4t + 4t^2 + 4t^4 + 8t^5 + 4t^8 + \cdots + 12t^{25} + \cdots$

The powers of t appearing in this expansion are precisely the exponents expressible as a sum of two squares.
The coefficient of t^n on the right hand side is the number of solutions of $n = x^2+y^2$ $(x,y\in\mathbb{Z})$

eg. $25 = x^2+y^2$ has 12 solutions $(\pm5, 0)$, $(0,\pm5)$, $(\pm3,\pm4)$, $(\pm4,\pm3)$
$5 = x^2+y^2$ has 8 solutions $(\pm2,\pm1)$, $(\pm1,\pm2)$
$6 = x^2+y^2$ has 0 solutions.

$\theta(t)^3 = 1 + 6t + 12t^2 + 8t^3 + 6t^4 + 24t^5 + 24t^6 + 12t^8 + \cdots$

$1 = x^2+y^2+z^2$ has six solutions $(\pm1,0,0)$, $(0,\pm1,0)$, $(0,0,\pm1)$
$2 = $ ... twelve $(\pm1,\pm1,0)$, $(\pm1,0,\pm1)$, $(0,\pm1,\pm1)$
$7 = x^2+y^2+z^2$ has 0 solutions

$\theta(t)^4$ has positive coefficient of t^n for every positive integer n
Lagrange's Theorem: every positive integer is a sum of four squares

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Fundamental Theorem of Arithmetic: Every positive integer is uniquely expressible as a product of primes.
We'll explain exactly what this says and we'll prove it.

Fundamental Theorem of Calculus
" " of Linear Algebra
" " " Algebra

FTA = Fundamental Theorem of Arithmetic:
Positive integers factor uniquely as products of primes.

eg. $12 = 6\times 2 = 2\times 3\times 2$

$12 = 3\times 4 = 3\times 2\times 2$

$12 = (-2)\times(-6) = 2\times 6$

$12 = \boxed{1}\times 12$

Ignore factors of ± 1 in factorization: these are <u>units</u>. (invertible elements). In $\mathbb{Z}$, the only units are ± 1.

eg. $\mathbb{Z}[i] = \{a+bi : a,b\in\mathbb{Z}\}$ (commutative ring with identity 1) we again have unique factorization. The units are $\pm 1, \pm i$.

$i\cdot(-i) = 1$

$1\cdot 1 = 1$

$(-1)(-1) = 1$

$\mathbb{Z}[i]$ = "Gaussian integers"

Elements of R:
- zero
- units
- irreducible
- reducible.

$12 = 2\times 2\times 3 = \underbrace{(1+i)(1-i)(1+i)(1-i)3}_{\text{cannot be factored any further; they are irreducible.}} = (i-1)(-i-1)(1+i)(1-i)3$

$2+i = i(1-2i)$

$10 = 2\times 5 = (1+i)(1-i)(1+2i)(1-2i) = (1+i)(1-i)(2+i)(2-i)$ "migration of units"

$\times i$, $\times(-i)$

A (commutative ring with identity) has unique factorization if every nonzero element can be factored as a product of irreducible elements and this factorization is essentially unique (i.e. unique up to permutation of factors and migration of units.

Why don't we just say primes and composites instead of irreducible and reducible elements?

In $\mathbb{Z}$:
- zero: 0
- units: ± 1
- irreducible: $\pm 2, \pm 3, \pm 5, \pm 7, \pm 11, \cdots$
- reducible: $\pm 4, \pm 6, \pm 8, \pm 9, \pm 10, \pm 12, \cdots$

In $\mathbb{Z}[i]$:
- zero: 0
- units: $\pm 1, \pm i$
- irreducibles: $\pm 1 \pm i = \{1+i, 1-i, -1+i, -1-i\}$ (actually $3, -3, 3i, -3i$)
  $\pm 1 \pm 2i, \pm 2 \pm i$ etc.
- reducibles: 2 (actually $2, -2, 2i, -2i$)

A nonzero element $\alpha \in R$ is reducible if $\alpha \neq 0$, $\alpha \neq$ unit and $\alpha = \alpha_1 \alpha_2$ α_1, α_2 not units.

α is irreducible if the only factorizations $\alpha = \alpha_1 \alpha_2$ have either α_1 or α_2 is a unit.

Eg. in $\mathbb{Z}[i]$, 3 is irreducible. If $3 = \beta\gamma$, $\beta, \gamma \in \mathbb{Z}[i]$ then either $\beta \in \{\pm 1, \pm i\}$ or $\gamma \in \{\pm 1, \pm i\}$

2 is reducible since $2 = (1+i)(1-i)$, neither $1+i$ nor $1-i$ is a unit.

Eg. the ring $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\}$ does not have unique factorization.

$\mathbb{Z}[\sqrt{6}] = \{a + b\sqrt{6} : a, b \in \mathbb{Z}\}$ . . . . . .

An example of non-unique factorization in $\mathbb{Z}[\sqrt{6}]$: $6 = 2 \times 3 = \sqrt{6} \times \sqrt{6}$

$2, 3, \sqrt{6}$ are irreducible in $\mathbb{Z}[\sqrt{6}]$.

But be careful: $\mathbb{Z}[\sqrt{6}]$ has infinitely many units.

Pell's equation $x^2 - 6y^2 = 1$ has solutions $(\pm 1, 0)$, $(\pm 5, \pm 2)$

$(5 + 2\sqrt{6})(5 - 2\sqrt{6}) = 1$ in $\mathbb{Z}[\sqrt{6}]$

$x^2 - 6y^2 = -1$ has no integer solutions.

(Look at the equation mod 3)

In $\mathbb{Z}[\sqrt{-5}]$, the only units are ± 1.

Unique factorization is not universal.

Difference between $\mathbb{Z}[i]$, $\mathbb{Z}$, $\mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}[\sqrt{-5}]$, $\mathbb{Z}[\sqrt{6}]$

$\mathbb{Z}[i]$: unique factorization

$\mathbb{Z}[\sqrt{-5}]$: non-unique factorization

$15 = 3\times 5 = (-3)\times(-5)$ in $\mathbb{Z}$

$15 = 3\times(2+i)(2-i)$ in $\mathbb{Z}[i]$

$= 3\times(1+2i)(1-2i)$

$(2-i)i = 1+2i$, $(2+i)(-i) = 1-2i$

migration of units

$15 = 3\times 5 = (3+3\sqrt{2})(-5+5\sqrt{2})$ in $\mathbb{Z}[\sqrt{2}]$

$\times(1+\sqrt{2})$ $\times(-1+\sqrt{2})$

$(1+\sqrt{2})(-1+\sqrt{2}) = 1$

Infinitely many units in $\mathbb{Z}[\sqrt{2}]$: $\pm(1+\sqrt{2})^k = \pm 1, \pm 1\pm\sqrt{2}, \pm 3\pm 2\sqrt{2}, \cdots$

$3^2 - 2\cdot 2^2 = (3+2\sqrt{2})(3-2\sqrt{2}) = 1$

Solutions of $(a+b\sqrt{2})(a-b\sqrt{2}) = 1$

$a^2 - 2b^2 = 1$

$6 = 2\times 3 = (1+\sqrt{-5})(1-\sqrt{-5})$

where all factors $2, 3, 1+\sqrt{-5}, 1-\sqrt{-5}$ are irreducible in $\mathbb{Z}[\sqrt{-5}]$

If holds in $\mathbb{Z}$, $\mathbb{Z}[i]$, $\mathbb{Z}[\sqrt{2}]$, $\mathbb{Z}[\sqrt{3}]$, $\mathbb{Z}[\sqrt{5}]$, $\mathbb{Z}[\sqrt{-2}]$, $\mathbb{Z}[\sqrt{-3}]$

Not in $\mathbb{Z}[\sqrt{-5}]$ or $\mathbb{Z}[\sqrt{6}]$.

In $\mathbb{Z}[\sqrt{-5}]$ the only units are ± 1. why

If x is a unit $x = a+b\sqrt{-5}$ then

$x\bar{x} = (a+b\sqrt{-5})(a-b\sqrt{-5}) = a^2+5b^2 = 1$

But the only integer solutions are $(a,b) = (\pm 1, 0)$.

$2, 3, 1+\sqrt{-5}, 1-\sqrt{-5} \in \mathbb{Z}[\sqrt{-5}]$ are irreducible.

why? If $2 = \alpha\beta$, $\alpha, \beta \in \mathbb{Z}[\sqrt{-5}]$ then

then $4 = |2|^2 = |\alpha|^2|\beta|^2$

$\alpha = a+c\sqrt{-5}$, $\beta = b+d\sqrt{-5}$

$|\alpha|^2 = (a+c\sqrt{-5})(a-c\sqrt{-5}) = a^2+5c^2$

$|\beta|^2 = b^2+5d^2$

$4 = |\alpha|^2|\beta|^2$ where $|\alpha|^2, |\beta|^2 \in \{1,2,3,4,5,\cdots\}$

$1\times 4 \Rightarrow \alpha$ unit

~~2×2~~ $a^2+5c^2 = 2$ has no solution.

$4\times 1 \Rightarrow \beta$ is a unit.

Why does $\mathbb{Z}$ have unique factorization?

It's rather easy to show every integer factors into primes.

$12 = 3 \times 4 = 3 \times 2 \times 2$

There is no infinite decreasing sequence $n_1 > n_2 > n_3 > n_4 > \cdots > 0$ in the positive integers.

The positive integers are well-ordered. (Equivalently, use induction).

Why is the prime factorization of a positive integer n unique?

Can $n = p_1 p_2 p_3 \cdots p_r = q_1 q_2 q_3 \cdots q_s$ where $p_1, p_2, \cdots, p_r, q_1, q_2, \cdots, q_s$ are primes?

We would need to show that every prime on the left (every p_i) occurs also among $q_1, \cdots, q_s$.

Basic step: If p is a prime and $a, b \in \mathbb{Z}$ with $p \mid ab$, then show $p \mid a$ or $p \mid b$.

Euclid's Lemma If a prime p divides ab in $\mathbb{Z}$, then $p \mid a$ or $p \mid b$.

("or" is inclusive. Eg. $3 \mid 6 \cdot 9 \Rightarrow 3 \mid 6$ or $3 \mid 9$.)

$6 \mid 4 \cdot 9$ but $6 \nmid 4$, $6 \nmid 9$. We really need p to be prime.

$p \mid ab$ says: $pk = ab$ for some k. Can we argue: p is a prime factor on the left so it's a factor on the right so p is a factor in a or in b. No! This is a common fallacy.

Proof of Euclid's lemma: Suppose $p \mid ab$ ie. $pk = ab$ for some k.
Either $p \mid a$ or $p \nmid a$. If $p \mid a$ we're done. Hence we may assume $p \nmid a$.

Then $\gcd(p, a) = 1 = rp + sa$ for some $r, s \in \mathbb{Z}$. ← p not in this list

Divisors of a: $\pm 1, \cdots, \pm a$

Divisors of p: $\pm 1, \pm p$.

$b = rbp + sab$ is divisible by p since both terms rbp and sab are divisible by p. □

If $p \mid abc$ where p is prime then $p \mid a$ or $p \mid b$ or $p \mid c$.

$p \mid abc \Rightarrow p \mid a$ or $p \mid bc$. $\Rightarrow p \mid a$ or $p \mid b$ or $p \mid c$.

This argument extends to any number of factors ie. if $p \mid q_1 q_2 \cdots q_l$ where p is prime then $p \mid q_j$ for some $j \in \{1, 2, \cdots, l\}$. (To formalize this argument, use induction.)

Fundamental Theorem of Arithmetic Every positive integer has a unique factorization as a product of primes.

If $n = p_1 p_2 \cdots p_r = q_1 q_2 \cdots q_s$ where $p_1, p_2, \cdots, p_r, q_1, q_2, \cdots, q_s$ are primes then $p_1 \mid n$ so $p_1 \mid q_j$ for some $j \in \{1, 2, \cdots, s\}$ so $p_1 = q_j$. Cancel this prime factor from both sides and repeat the argument with the remaining prime factors.

On to Chapter 9.

Look at powers in $\mathbb{Z}/7\mathbb{Z} = \mathbb{F}_7 = \{0,1,2,3,4,5,6\} = \{0,\pm1,\pm2,\pm3\}$ eg. $3^2 = 9 = 2$, $\frac{1}{4} = 2$.

$-3 = 6+6+6 = 4$

x	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8
0	1	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1
2	1	2	4	1	2	4	1	2	4
3	1	3	2	6	4	5	1	3	2
$-3=4$	1	4	2	1	4	2	1	4	2
$-2=5$	1	5	4	6	2	3	1	5	4
$-1=6$	1	6	1	6	1	6	1	6	1

$x^5 = x^{-1} = \frac{1}{x}$ if $x \neq 0$

palindromic sequence $1,4,2,2,4,1$

Theorem Let p be prime. Then for all $a \in \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} = \{0,1,2,\dots,p-1\}$, $a^p = a$;
furthermore $a^{p-1} = 1$ if $a \neq 0$.

Remark We really do need p to be prime.

"Fermat's Little Theorem" can be rephrased as:
Let p be prime. Then for all $a \in \mathbb{Z}$, $a^p \equiv a \bmod p$. Moreover if $p \nmid a$ (p doesn't divide a) then $a^{p-1} \equiv 1 \bmod p$.

Solve the linear system $\begin{cases} 3x + 5y = 2 \\ 4x + 7y = 5 \end{cases}$ for $x, y \in \mathbb{F}_{11} = \{0, 1, 2, \cdots, 10\}$

$\frac{1}{3} = 4$

$$\begin{matrix} x & y \\ \end{matrix}$$
$$\left[\begin{array}{cc|c} 3 & 5 & 2 \\ 4 & 7 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 9 & 8 \\ 4 & 7 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 9 & 8 \\ 0 & 4 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 9 & 8 \\ 0 & 1 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 7 \end{array}\right]$$
$$\begin{matrix} & & x & y \\ \end{matrix}$$

$7 - 36 = 7 - 3 = 4$

$5 - 32 = 5 + 1 = 6$

$8 - 63 + 66 = 8 + 3 = 0$

The unique solution is $(x, y) = (0, 7)$.

Check: $(0, 7)$ satisfies both equations.

Computing a^p and a^{p-1} may be infeasible but computing $a^p \bmod p$ and $a^{p-1} \bmod p$ is feasible even when a, p are hundreds of digits long.

Mathematica commands `Mod[a^k, n]` computes $a^k \bmod n$. This will fail if a, k are hundreds of digits long. Instead use the command `PowerMod[a, k, n]`

"Modular exponentiation" is a very important operation in cryptography, primality testing, pseudorandom number generation, etc.

Factorization of integers is much harder than primality testing.

(believed to be) non-poly. time. — infeasible

polynomial-time — feasible

Fermat's Little Theorem Let $\mathbb{F}_p = \{0,1,2,\cdots,p-1\} = \mathbb{Z}/p\mathbb{Z}$ where p is prime. Then $a^p = a$ for all $a \in \mathbb{F}_p$; and $a^{p-1} = 1$ for all nonzero $a \in \mathbb{F}_p$.

Proof Consider the product of all nonzero elements of $\mathbb{F}_p$: $b = \prod_{0 \neq a \in \mathbb{F}_p} a = 1 \times 2 \times 3 \times \cdots \times (p-1) \in \mathbb{F}_p^\times = \{1,2,\cdots,p-1\}$
$= \{$units in $\mathbb{F}_p\}$
$= \{$nonzero elements in $\mathbb{F}_p\}$ since p is prime

(Actually $b = (p-1)!$ but this is now mod p.) Let $a \in \mathbb{F}_p^\times$ (nonzero element of $\mathbb{F}_p$).
The map $x \mapsto ax$, $\mathbb{F}_p \to \mathbb{F}_p$ is bijective. (Its inverse is $x \mapsto a^{-1}x$).

$b = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (p-1)$
$a^{p-1}b = a \cdot 2a \cdot 3a \cdot \cdots \cdot (p-1)a = b \Rightarrow a^{p-1}bb^{-1} = bb^{-1}, \Rightarrow a^{p-1} = 1.$

eg. $p=7$, $a=3$, $b = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6$
$3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18$
$3 \cdot 6 \cdot 2 \cdot 5 \cdot 1 \cdot 4 = b$

For the first conclusion, let $a \in \mathbb{F}_p$ (zero or nonzero).
If $a \neq 0$ then $a \cdot a^{p-1} = 1 \cdot a$ so $a^p = a$. If $a = 0$ then $a^p = 0 = a$. □

Fermat's Little Theorem is really a special case of Lagrange's theorem in group theory.
If H is a subgroup of a finite group G then $|H|$ divides $|G|$. In particular the order of every element of G divides $|G|$. So $g^n = 1$ for all $g \in G$ where $n = |G|$.
$\mathbb{F}_p^\times$ is a group of order $p-1$ so $a^{p-1} = 1$ for all $a \in \mathbb{F}_p^\times$.

Fermat's Little Theorem plays a huge role in primality testing.

Generalization of Fermat's Little Theorem (Euler's Formula/Theorem) allows us to work with modular exponentiation mod m, m not necessarily prime.

This requires Euler's "totient" function $\phi(m)$ = number of integers $k \in \{1, 2, \cdots, m\}$ that are relatively prime to m

"Freud"

eg. $\phi(p) = p-1$ whenever p is prime

m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\phi(m)$	1	1	2	2	4	2	6	4	6	4	10	4	12	6	8	8	16

$\phi(5) = 4$

$\phi(25) = 20 \neq \phi(5)\phi(5)$

$\phi(9) \neq \phi(2)\phi(2)$

1, 2, 3, 4, 5, 6

1, 2, 3, 4, 5

1, 2, 3, 4

$\phi(p^k) = p^k - p^{k-1} = p^k(1 - \frac{1}{p})$ (p prime, $k \geq 1$)

$\phi(mn) = \phi(m)\phi(n)$ whenever $\gcd(m, n) = 1$.

$\mathbb{Z}/4\mathbb{Z} = \{0, 1, 2, 3\}$

mod 2

	0	1
0	0	
1		1

mod 5

	0	1	2	3	4
0	0	6	12	3	9
1	10	1	7	13	4
2	5	11	2	8	4

mod 3

$\mathbb{Z}/15\mathbb{Z} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$

$\cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$

$\phi(15) = 8 = \phi(3)\phi(5)$

$\phi(30) = \phi(5)\phi(6) = 4 \cdot 2 = 8$

$\mathbb{Z}/30\mathbb{Z} = \{0, 1, 2, \cdots, 29\}$

$\{k \in \mathbb{Z}/30\mathbb{Z} : \gcd(k, 30) = 1\}$

$= \{1, 7, 11, 13, 17, 19, 23, 29\}$

mod 6

	0	1	2	3	4	5
0	0	25	20	15	10	5
1	6	1	26	21	16	11
2	12	7	2	27	22	17
3	18	13	8	3	28	23
4	24	19	14	9	4	29

mod 5

If R, S are rings then $R \oplus S$ (also denoted $R \times S$) is $R \oplus S = \{(r, s) : r \in R, s \in S\}$ is a ring with componentwise operations

$(r, s) + (r', s') = (r+r', s+s')$

$(r, s)(r', s') = (rr', ss')$

$(r, s) \in R \oplus S$ is a unit iff r is a unit in R and s " " " S.

For example we look for integer solutions of some Diophantine equation e.g. $61x^2 - y^2 = 31$. Does this have integer solutions? If so, show by example. If not (i.e. no solutions in $\mathbb{Z}$) then find $m \in \mathbb{Z}$ such that the equation has no solutions in $\mathbb{Z}/m\mathbb{Z}$. In this case it's best to consider prime powers $m = p^k$, p prime, $k \geq 1$.

$x^2 - 3y^2 = 3$ has solutions mod 2 but no solutions mod 4. (So no solutions in $\mathbb{Z}$)
$x^2 + y^2 + z^2 = 7$ has solutions mod 4 but no solutions mod 8. (- - - - -)

Euler's Formula If a, m are integers, $m \geq 1$ and $\gcd(a,m) = 1$, then $a^{\phi(m)} \equiv 1 \bmod m$.

This generalizes Fermat's Little Theorem: if p is prime and $\gcd(a,p) = 1$ then $a^{\phi(p)} \equiv 1 \bmod p$ so $a^{p-1} \equiv 1 \bmod p$.

The proof of Euler's Formula is the same as the proof of Fermat's Little Theorem.
Both are special cases of Lagrange's Theorem for finite groups.

Eg. Find the last two decimal digits of 1234567^{531}

i.e. find $r \in \{0, 1, 2, \ldots, 99\}$ such that $1234567^{531} \equiv r \bmod 100$.

$1234567^{531} \equiv 67^{531} \bmod 100$

67 is relatively prime to 100 so $67^{\phi(100)} \equiv 1 \bmod 100$ where $\phi(100) = \phi(2^2 \cdot 5^2) = \phi(2^2)\phi(5^2)$

$= (2^2 - 2)(5^2 - 5) = 2 \times 20 = 40$ so $67^{40} \equiv 1 \bmod 100$ $\qquad 531 = 13 \times 40 + 21$

$$67^{531} = 67^{13 \times 40 + 21} = \underbrace{(67^{40})^{13}}_{1} \cdot 67^{21} \equiv 67^{21} \bmod 100$$

$67^{21} \equiv ?$ mod 100

$67^1 \equiv 67$

square → $67^2 \equiv 89$ mod 100 $\qquad 67^2 = 4489 \equiv 89$ mod 100

square → $67^4 \equiv 89^2 \equiv 21$ ...

square → $67^8 \equiv 21^2 \equiv 41$ ...

square → $67^{16} \equiv 41^2 \equiv 81$ ....

$67^{21} = 67^{16} \cdot 67^4 \cdot 67^1 \equiv 81 \cdot 21 \cdot 67 \equiv 67$ mod 100

So $1234567^{531} \equiv 67$ mod 100.

decimal 2025 written in binary?

2 | 2025		
2 | 1012	r 1	
2 | 506	r 0	
2 | 253	r 0	
2 | 126	r 1	
2 | 63	r 0	
2 | 31	r 1	
2 | 15	r 1	
2 | 7	r 1	
2 | 3	r 1	
2 | 1	r 1	
0	r 1	

$2025_{10} = 11111101001_{two}$

Fast exponentiation using binary representation of the exponent: $21 = 16 + 4 + 1$

i.e. the decimal number 21 is 10101

16's, 8's, 4's, 2's, 1's

i.e. $2025 = 1024 + 512 + 256 + 128 + 64 + 32 + 8 + 1$

$x^{2025} = x^{1024} \cdot x^{512} \cdot x^{256} \cdot x^{128} \cdot x^{64} \cdot x^{32} \cdot x^8 \cdot x^1$

Computing a^k mod m can be done in polynomial time.

See Chapter 16: ... repeated squaring.

Chapter 11: Euler's Phi Function and the Chinese Remainder Theorem

mod 5

mod 3	0	1	2	3	4
0	0	6	12	3	9
1	10	1	7	13	4
2	5	11	2	8	14

Solving a pair of congruences

If $\gcd(m,n)=1$ then the system $\begin{cases} x \equiv a \bmod m \\ x \equiv b \bmod n \end{cases}$ has a unique solution mod mn.

eg. solve $\begin{cases} x \equiv 1 \bmod 3 \\ x \equiv 3 \bmod 5 \end{cases}$ $\Leftrightarrow$ $x \equiv 13 \bmod 15$.

$\Leftarrow$ is trivial. Given $x \equiv 13 \bmod 15$, we compute

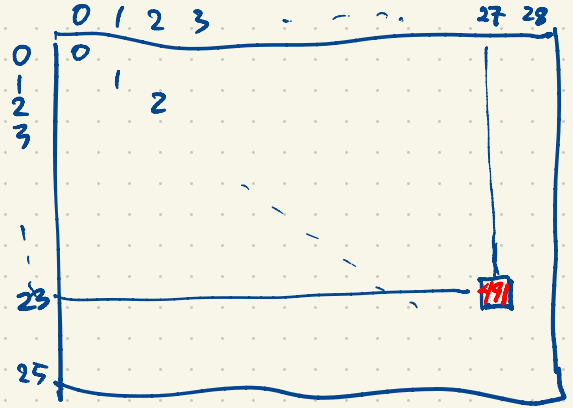
$x \equiv 13 \equiv 1 \bmod 3$

$x \equiv 13 \equiv 3 \bmod 5$.

$\Rightarrow$ takes a little more work.

eg. $m = 26$, $n = 29$ Solve $\begin{cases} x \equiv 23 \bmod 26 \\ x \equiv 27 \bmod 29 \end{cases}$ $\Leftrightarrow$ $x \equiv 491 \bmod 754$, $754 = 26 \cdot 29$

(checked that both the original congruences hold)



Answer without completing all 754 entries in the table.

$x \equiv 23 \bmod 26$

$x = 26k + 23 \equiv 27 \bmod 29$

$26k \equiv 4 \bmod 29$

$k \equiv -10 \cdot 26k \equiv -10 \cdot 4 \equiv -40 \equiv -11 \equiv 18 \bmod 29$ ($-10 \cdot 26 \equiv 1 \bmod 29$)

$k = 29r + 18$

$x = 26\underbrace{(29r+18)}_{k} + 23 = 754r + 491$

Find the inverse of 26 mod 29. ie. $-10 \equiv 19$

29	26	
1	0	29
0	1	26
1	−1	3
−8	9	2
9	−10	1

$\gcd(29,26) = 1 = 9 \cdot 29 - 10 \cdot 26$

Ignore this if you have already done the survey in another class

OR

www.uwyo.edu/mathstats

Let us know how you learn in mathematics classes! Please click the button below to access the survey. Thanks!

View of Learning Mathematics 2025 Survey

Chapter 12: Primes

An integer $p > 1$ is prime if its only divisors are $\pm 1, \pm p$.

Primes: $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, \ldots$

There are infinitely many primes! Euclid's proof: (by contradiction)

Suppose there are only finitely many primes $p_1, p_2, p_3, \ldots, p_n$ ($p_1 = 2$, $p_2 = 3$, $p_3 = 5$, p_n = largest prime)

To get a contradiction, consider $A = p_1 p_2 p_3 \cdots p_n + 1$, which is bigger than every prime so it isn't prime so A has a prime factor p with $1 < p < A$. Then $p \in \{p_1, \ldots, p_n\}$ by hypothesis so $p \mid p_1 p_2 \cdots p_n$ and $p \mid A$ so $p \mid (A - p_1 p_2 \cdots p_n)$, a contradiction.

There are other proofs that there are infinitely many primes including Euler's proof which actually shows more: $\sum_{p \text{ prime}} \frac{1}{p}$ diverges i.e. $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \frac{1}{17} + \cdots$ diverges.

Compare: the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \quad \text{diverges.}$$

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots \quad \text{converges to 1.}$$

Theorem The sum of the reciprocals of the twin primes converges:

$$\left(\tfrac{1}{3} + \tfrac{1}{5}\right) + \left(\tfrac{1}{5} + \tfrac{1}{7}\right) + \left(\tfrac{1}{11} + \tfrac{1}{13}\right) + \left(\tfrac{1}{17} + \tfrac{1}{19}\right) + \left(\tfrac{1}{29} + \tfrac{1}{31}\right) + \cdots < \infty$$

The sequence of powers of 2: $1, 2, 4, 8, 16, 32, 64, \ldots$ has unbounded gaps

The sequence of primes also has unbounded gaps

Proof that there are gaps > 1000000 in the sequence of primes: We construct a large positive integer N such that $\{N - 1000000, N - 999999, \ldots, N-2, N-1, N\}$ contains no primes. Take $N = 1000001! + 1000001$

$N = \underbrace{1000001!}_{1\times 2\times 3\times\cdots\times 1000001} + 1000001$ is divisible by 1000001

$N-1 = 1000001! + 1000000$ ---------- 1000000

$N-2 = 1000001! + 999999$ ---------- 999999

$N-3 = 1000001! + 999998$ ---------- 999998

$\vdots$

$N-999999 = 1000001! + 2$ ---------- 2

None of $N-99999, N-999998, \cdots, N$ are prime.

	M	W	F
Mar	24	26	28
	31	2	4
Apr	7	9	11

Test on Ch 1-12 (material before Spring Break) 7:30am Mon Apr 7.
Expect (i) summary/review sheet and (ii) practice problems this week.

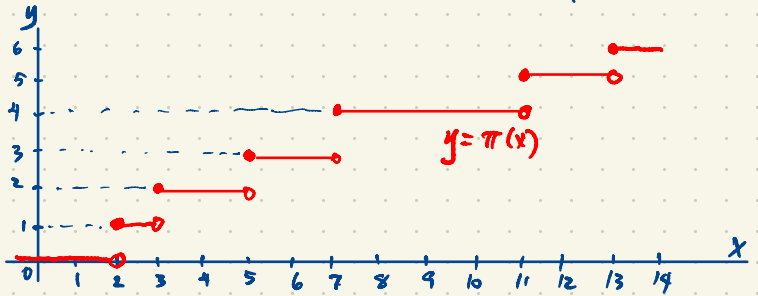
Chapter 13: Counting Primes

$\pi(x)$ = number of primes $\leq x$

eg. $\pi(10) = 4$ since there are exactly four primes ≤ 10, namely 2, 3, 5, 7

$\pi(10.99999) = 4$

$\pi(11) = 5$ (there are five primes ≤ 11, namely 2, 3, 5, 7, 11)



The number of primes with at most two decimal digits is $\pi(100) = 25$ (p.92)

The number of primes with exactly three digits is $\pi(1000) - \pi(100) = 168 - 25 = 143$.

The number of primes in $(a,b]$ is

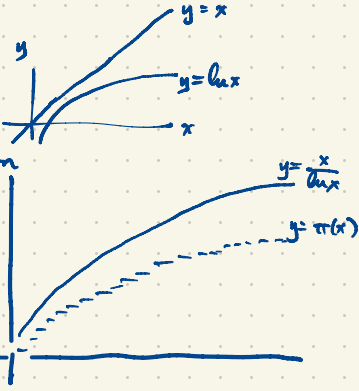
$\pi(b) - \pi(a)$

↑ no. of primes in $[0,b]$ ↑ no. of primes in $[0,a]$

How fast does $\pi(x)$ grow as $x \to \infty$? Slower than linear:

$$\lim_{x\to\infty} \frac{\pi(x)}{x} = 0$$

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$$



Prime Number Theorem $\pi(x) \sim \frac{x}{\ln x}$ "asymptotic"

Proof uses $\zeta(x)$ = Riemann zeta function
Independently proved by Hadamard and de la Vallée Poussin 1896
Selberg and Erdős gave more elementary proofs 1949

i.e. $\lim_{x\to\infty} \frac{\pi(x)}{x/\ln x} = 1$ i.e. $\frac{\pi(x)}{x/\ln x} \to 1$ as $x \to \infty$.

$\pi(10^{29})$ = how many primes have at most 29 decimal digits

= 1 520 698 109 714 272 166 094 258 063

$\frac{10^{29}}{\ln 10^{29}}$ = 1 513 829 040 151 813 892 012 209 509. ....

$Li(10^{29})$ = 1 520 698 109 714 276 717 287 880 527. ...

Alternative form of PNT (Prime Number Theorem):

For N large, the number of primes in $(N, N+\Delta N]$ is ΔN smaller than N is

$$\pi(N+\Delta N) - \pi(N) \approx \frac{\Delta N}{\ln N}$$

i.e. for numbers $n \approx N$ chosen randomly, the probability that n is prime is about $\frac{1}{\ln N}$.

For random 100-digit numbers, the probability of being prime is $\frac{1}{\ln 10^{100}} \approx 0.00434$ i.e. 0.4%

. . . . . odd numbers, ...

$\frac{2}{\ln 10^{100}} \approx 0.0087$ almost 1%.

This leads to a better estimate for $\pi(x) \approx Li(x) = \int_2^x \frac{dt}{\ln t}$

$$\pi(x) \sim Li(x) \sim \frac{x}{\ln x}$$

$\sqrt{x^2+2x} \sim x$ as $x \to \infty$

Since $\frac{\sqrt{x^2+2x}}{x} = \sqrt{1+\frac{2}{x}} \to 1$ as $x \to \infty$.

but $\sqrt{x^2+2x} - x \to 1$.

$\sqrt{x^6+x^5} \sim x^3$ as $x \to \infty$

$\frac{\sqrt{x^6+x^5}}{x^3} \to 1$ as $x \to \infty$

But $\sqrt{x^6+x^5} - x^3 \to \infty$ as $x \to \infty$.

googol = 10^{100}

The Euler totient function $\phi(n)$ = the number of integers k, $1 \le k \le n$, satisfying $\gcd(k, n) = 1$

This function is multiplicative: $\phi(mn) = \phi(m)\phi(n)$ whenever $\gcd(m, n) = 1$.

eg. $\phi(15) = \phi(3)\phi(5)$ since $\gcd(3, 5) = 1$.

$8 = 2 \times 4$

1, 2, 4, 7, 8, 11, 13, 14 relatively prime with 15

1, 2 rel. prime to 3

1, 2, 3, 4 - - - - - 5

$\phi(p) = p - 1$ whenever p is prime

$\phi(32) \neq \phi(4)\phi(8)$

$16 \neq 2 \cdot 4$ ($\phi(4) = 2$, $\phi(8) = 4$)

1, 3 rel. prime to 4

1, 3, 5, 7 - - - - - 8

1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31 rel. prime to 32

If $f(mn) = f(m)f(n)$ for all m, n integers, we say f is completely multiplicative.

Next: some more examples of multiplicative functions.

$\sigma(m)$ = sum of the positive integer divisors of m

eg. $\sigma(12) = 1+2+3+4+6+12 = 28 =$

$\sigma(10) = 1+2+5+10 = 18 = (1+2)(1+5) = \sigma(2)\sigma(5)$

$\sigma(11) = 1+11 = 12$

$\sigma(p) = p+1 \iff p$ is prime

$12 = 4 \cdot 3$

$\sigma(12) = (1+2+4)(1+3) = \sigma(4)\sigma(3)$

σ is a multiplicative function: $\sigma(mn) = \sigma(m)\sigma(n)$ whenever $\gcd(m,n) = 1$.

$12 = 2 \cdot 6$ but $28 = \sigma(12) \neq \sigma(2)\sigma(6) = (1+2)(1+2+3+6) = 3 \cdot 12 = 36$

Of course the sum of all divisors of 6 is $(-6)+(-3)+(-2)+(-1)+1+2+3+6 = 0$

The perfect number problem: When can the sum of the proper positive divisors of n be equal to n? Historically such numbers were called perfect. eg. $6 = 1+2+3$ is perfect. n is perfect iff $\sigma(n) = 2n$.

$\sigma(6) = \underbrace{1+2+3}_{6} + 6 = 2 \cdot 6 = 12$.

28 is also perfect: $28 = 1+2+4+7+14$

$$\sigma(28) = \underbrace{1+2+4+7+14}_{28} + 28 = 2\cdot 28 = 56.$$

$28 = 4\cdot 7$

$\sigma(28) = \sigma(4)\sigma(7) = (1+2+4)(1+7) = 7\cdot 8 = 56.$

The smallest perfect numbers are $6, 28, 496 = 16\cdot 31$

$\sigma(496) = \sigma(16)\sigma(31) = (1+2+4+8+16)(1+31) = 31\cdot 32 = 2\cdot 496$

$$\begin{array}{r} 31 \\ 16 \\ \hline 186 \\ 31 \\ \hline 496 \end{array}$$

Open questions:

- Are there any odd perfect numbers?
- Are there infinitely many perfect numbers?

$\sigma(p) = p+1$ whenever p is prime

$\sigma(p^k) = 1+p+p^2+\cdots+p^k = \frac{p^{k+1}-1}{p-1}$ since $(p-1)(1+p+p^2+\cdots+p^k) = -1+p-p+p^2-p^2+\cdots+p^k-p^{k-1}+p^{k+1} = p^{k+1}-1$

$\sigma(16) = \sigma(2^4) = \frac{2^5-1}{2-1} = 31$

One way (maybe the only way?) to get a perfect number is to find a prime number p such that 2^p-1 is prime and then take $n = 2^{p-1}(2^p-1)$

$$\sigma(n) = \sigma(2^{p-1})\,\sigma(2^p-1) = \frac{2^p-1}{2-1}\left(2^{p-1}+1\right) = \left(2^p-1\right)2^p = 2n$$

How often is 2^p-1 a prime number?

Denote $M_p = 2^p-1$.

p	M_p	
1	$2^1-1=1$	not prime
2	$2^2-1=3$	is prime
3	$2^3-1=7$	is prime
4	$2^4-1=15$	is not prime
5	$2^5-1=31$	is prime
6	$2^6-1=63$	is not prime
7	$2^7-1=127$	is prime
8	$2^8-1=255$	is not prime
9	$2^9-1=511$	is not prime
10	$2^{10}-1=1023$	is not prime
11	$2^{11}-1=2047=23\times 89$	is not prime.

$2^{ab}-1$ is divisible by 2^a-1

$$x^k-1 = (x-1)(1+x+x^2+\cdots+x^{k-1})$$

Let $x=2^a$, $k=b$.

$$2^{ab}-1 = (2^a-1)(1+2^a+2^{2a}+\cdots+2^{(b-1)a})$$

giving a nontrivial factorization of $2^{ab}-1$ if $a,b>1$.

In order for a number of the form $M_p = 2^p-1$ to be prime, we require p to be prime.

However the converse fails: if p is prime, it does not follow that $M_p = 2^p-1$ is prime.

Primes of the form $M_p = 2^p - 1$ (p prime) are called Mersenne primes.

Only 52 Mersenne primes are known : see GIMPS

Theorem Let n be an even integer. Then n is perfect iff $n = 2^{p-1}(2^p - 1)$ where p and $2^p - 1$ are prime. Thus we have a one-to-one correspondence between even perfect numbers and Mersenne primes

$$n = 2^{p-1}(2^p - 1) \longleftrightarrow M_p = 2^p - 1$$

In particular exactly 52 perfect numbers are known.
We don't know whether there are infinitely many Mersenne primes and we don't know whether there are any odd perfect numbers.

Proof of the theorem. In one direction we have seen that if p and $M_p = 2^p - 1$ are prime then $\sigma(n) = 2n$.

Conversely, suppose n is an even perfect number; we must show that $n = 2^{p-1}(2^p - 1)$ where both p and $2^p - 1$ are prime. Now $n = 2^k m$ where $k \geq 1$ and m is odd. Since n is perfect, $2^{k+1} m = 2n = \sigma(n) = \sigma(2^k)\sigma(m) = \frac{2^{k+1}-1}{2-1}\sigma(m)$
$= (2^{k+1} - 1)\sigma(m)$
where $2^{k+1} - 1$ is odd, so $2^{k+1} \mid \sigma(m)$. (Euclid's Lemma)

Now $\sigma(m) = 2^{k+1}c$ for some positive integer c and

$$2^{k+1}m = (2^{k+1}-1)\sigma(m) = (2^{k+1}-1)2^{k+1}c \Rightarrow m = (2^{k+1}-1)c.$$

If $c>1$ then $2^{k+1}c = \sigma(m) \geq \underbrace{1+c+m}_{\text{distinct divisors of } m} = 1+c+(2^{k+1}-1)c = 1+2^{k+1}c$, a contradiction.

so $c=1$, $m = 2^{k+1}-1$,

$\sigma(m) = 1+m$, m is prime hence a Mersenne prime so $k+1 = p$ is prime and $n = 2^k m = 2^{p-1}(2^p-1)$. □

The largest known primes are Mersenne primes. (Here we're looking just at the explicitly known primes.) This is likely to be the case in the future as more primes are discovered. Mersenne numbers $M_n = 2^n-1$ are easier to check for primality than other numbers. The lamplighter effect.

What are large primes good for? (The largest Mersenne primes, ie. the largest known primes, are currently tens of millions of decimal digits long.) The largest primes used in cryptographic applications currently, are only a few hundred digits long, possibly a few thousand digits if you're paranoid.

$\mathbb{F}_p = \{0, 1, 2, \cdots, p-1\}$ where p is prime

Eg. $\mathbb{F}_7 = \{0, 1, 2, 3, 4, 5, 6\}$

$\mathbb{R}$ is an ordered field; $\mathbb{F}_7$ is not an ordered field.

$\mathbb{C}$ is not an ordered field.

Is $i > 0$ or $i < 0$?

Pos × Pos = Pos Neg × Neg = Pos

$i \times i = -1 > 0$ $i \times i = -1$

Instead of "positive" and "negative" in $\mathbb{F}_7$, we talk about (nonzero) squares and nonsquares

1, 2, 4 3, 5, 6

a	a^2
0	0
1	1
2	4
3	2
$-3 = 4$	2
$-2 = 5$	4
$-1 = 6$	1

a	a^2
0	0
± 1	1
± 2	4
± 3	2

Multiplication Table

	0	1	2	4	3	5	6
0	0	0	0	0	0	0	0
1	0	1	2	4	3	5	6
2	0	2	4	1	6	3	5
4	0	4	1	2	5	6	3
3	0	3	6	5	2	1	4
5	0	5	3	6	1	4	2
6	0	6	5	3	4	2	1

Ignoring zero,

(i) Square × Square = Square

(ii) Nonsq × Nonsq = Square

(iii) Sq × Nonsq = Nonsq

(iv) Nonsq × Sq = Nonsq

These rules hold in $\mathbb{F}_p$ whenever p is an odd prime. Why?

(i) is obvious since $a^2 \times b^2 = (ab)^2$

(iii) follows by process of elimination; Similarly (iv).

(ii) — — — — — —

In $\mathbb{R}$, pos + pos = pos.
pos × pos = pos

In $\mathbb{F}_7$, sq × sq = sq

sq + sq = sq

1 + 2 = 3
sq sq nonsq

If p is an odd prime then $\mathbb{F}_p$ has $p-1$ nonzero elements ($1, 2, \cdots, p-1$), half of which are squares and the other half are nonsquares.

There are $\frac{p-1}{2}$ (nonzero) squares and $\frac{p-1}{2}$ nonsquares.

Eg. $p=11$

a	a^2
0	0
± 1	1
± 2	4
± 3	9
± 4	5
± 5	3

(nonzero) squares: 1, 3, 4, 5, 9

Nonsquares: 2, 6, 7, 8, 10

$p=5$

a	a^2
0	0
± 1	1
± 2	4

(Nonzero) squares: 1, 4

nonsquares: 2, 3

$p=3$:

a	a^2
0	0
± 1	1

(Nonzero) squares: 1

nonsquares: 2

$p=13$:

a	a^2
0	0
± 1	1
± 2	4
± 3	9
± 4	3
± 5	12
± 6	10

Nonzero squares: 1, 3, 4, 9, 10, 12

Nonsquares: 2, 5, 6, 7, 8, 11

Use Fermat's Little Theorem: Let p be an odd prime. For all $a \in \mathbb{F}_p$, $a^p = a$

In other words, the polynomial $f(x) = x^p - x$ has every element of $\mathbb{F}_p$ as a root. $a^{p-1} = 1$ if $a \neq 0$

For $p=7$, $f(x) = x^7 - x$ has 7 distinct roots in $\mathbb{F}_7$.

$$= \underbrace{x}_{x-0}(x-1)(x-2)(x-3)(x-4)(x-5)(x-6)$$

$$f(x) = x(x^6-1) = x(x^3+1)(x^3-1)$$

root: 0 — roots 3, 5, 6 (nonsquares) — roots 1, 2, 4 (squares)

Theorem Let p be an odd prime. Then

$$x^p - x = x(x^{p-1}-1) = x\left(x^{\frac{p-1}{2}}-1\right)\left(x^{\frac{p-1}{2}}+1\right) = x(x-1)(x-2)\cdots(x-(p-1))$$

Roots: all (nonzero) squares in $\mathbb{F}_p$

Roots: all nonsquares in $\mathbb{F}_p$

Every $a \in \mathbb{F}_p$ is a root of $x^p - x$. (Fermat's Little Theorem)

Proof Let $a \in \mathbb{F}_p$, $a \neq 0$, so a^2 is a nonzero square in $\mathbb{F}_p$. Then a^2 is a root of $x^{\frac{p-1}{2}}-1$ since $(a^2)^{\frac{p-1}{2}} - 1 = a^{p-1} - 1 = 0$ by Fermat's Little Theorem.
So by process of elimination, $x^{\frac{p-1}{2}}+1$ has as its $\frac{p-1}{2}$ roots all the nonsquares in $\mathbb{F}_p$.

This gives a criterion for checking when an element $a \in \mathbb{F}_p$ is a square or a nonsquare. called Euler's Criterion: given $a \in \mathbb{F}_p$, p an odd prime,

$$a^{\frac{p-1}{2}} \equiv \begin{cases} 1 & \text{if } a \text{ is square nonzero} \\ -1 & \text{" " " nonsquare} \\ 0 & \text{if } a = 0. \end{cases} \Bigg\} \bmod p.$$

Eg. is 7 a square or a nonsquare in $\mathbb{F}_{13}$? Use Euler's Criterion: $7^6 \equiv 10^3 \equiv 1000 \equiv 12 \equiv -1 \bmod 13$
So 7 is a nonsquare mod 13.

Definition Let p be an odd prime, $a \in \mathbb{Z}$. Then the Legendre Symbol $\left(\frac{a}{p}\right) \in \{-1, 0, 1\}$ is defined by $\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is nonzero square mod } p; \\ -1 & \text{if " " nonsquare mod } p; \\ 0 & \text{if } a \equiv 0 \bmod p. \end{cases}$

So $a^{\frac{p-1}{2}} \equiv \left(\frac{a}{p}\right) \bmod 13$.

eg. $\left(\frac{7}{13}\right) = -1$, $\left(\frac{10}{13}\right) = 1$, $\left(\frac{0}{13}\right) = 0$, $\left(\frac{39}{3}\right) = 0$, $\left(\frac{20}{13}\right) = -1$

$\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$ for all $a, b \in \mathbb{Z}$ (they don't need to be relatively prime)

The Legendre symbol $\left(\frac{a}{p}\right)$ is a completely multiplicative function of a (where the odd prime p is fixed)

eg. $\left(\frac{20}{13}\right) = \left(\frac{4}{13}\right)\left(\frac{5}{13}\right) = -1$ so $20 \equiv 7 \bmod 13$ is a nonsquare mod 13.

$\underbrace{\quad}_{-1} \quad \underbrace{\quad}_{+1} \quad \underbrace{\quad}_{-1}$

When is $-1 \equiv +1$ a square mod p? -1 is a { nonsquare mod $3, 7, 11, \ldots, 2003, \ldots$; square mod $5, 13$

Is -1 a square or a nonsquare mod $p = 2003$? Nonsquare.

Theorem Let p be an odd prime. Then $\left(\frac{p-1}{p}\right) = \left(\frac{-1}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \bmod 4 \\ -1 & \text{if } p \equiv 3 \bmod 4. \end{cases}$

The quick proof uses $\left(\frac{-1}{p}\right) \equiv (-1)^{\frac{p-1}{2}} \bmod p$

If $p = 4k+1$ then $\frac{p-1}{2} = 2k$, $(-1)^{2k} = 1$.

If $p = 4k+3$ then $\frac{p-1}{2} = 2k+1$, $(-1)^{2k+1} = -1$.

How hard is it to find a nonsquare mod p (given an odd prime p)?