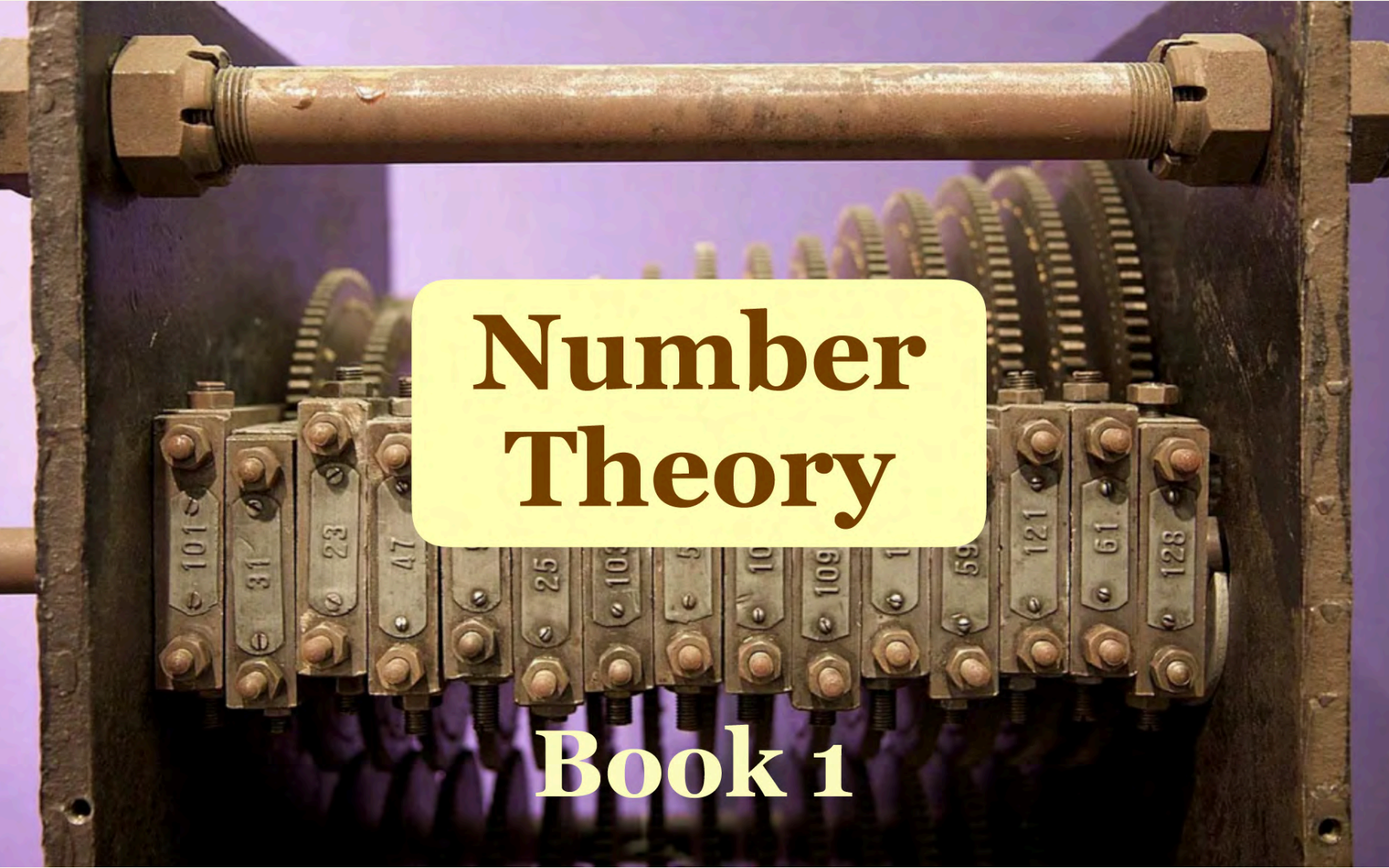
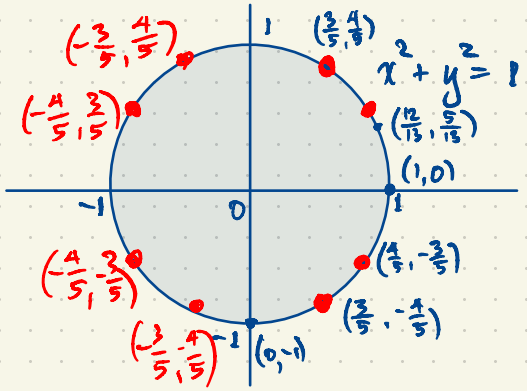

Number
Theory
Book 1



How many points on the circle $x^2+y^2=1$ $(x,y\in\mathbb{Q})$ have rational number coordinates?

Not $(\frac{1}{2},\pm\frac{\sqrt{3}}{2})$

Are there infinitely many "rational points" on the unit circle?

$(\frac{3}{5},\frac{4}{5}) \longleftrightarrow 3^2+4^2=5^2$ solution of $x^2+y^2=z^2$ $(x,y,z\in\mathbb{Z})$

A Pythagorean triple is a triple (a,b,c) of positive integers a,b,c, satisfying $a^2+b^2=c^2$.

eg. $(3,4,5)$, $(6,8,10)$, $(9,12,15)$, $(5,12,13)$, ...

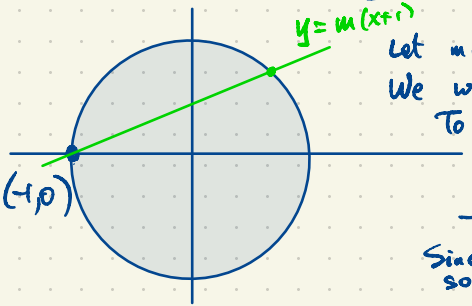
A triple (a,b,c) is primitive if it is not an integer scalar multiple of a smaller triple eg. $(3,4,5)$ is primitive; $(6,8,10)=2(3,4,5)$ is imprimitive, as is $(9,12,15)=3(3,4,5)$.

There are infinitely many primitive Pythagorean triples.

The triple $(3,4,5)$ yields eight rational points $(\pm\frac{3}{5},\pm\frac{4}{5})$, $(\pm\frac{4}{5},\pm\frac{3}{5})$. So does $(4,3,5)$

Theorem There are infinitely many rational points on the unit circle $x^2+y^2=1$. See Chapter 3.

Proof



Let $m\in\mathbb{Q}$. Consider the line $y=m(x+1)$ through $(-1,0)$.

We will see that this line intersects the circle in two rational points.

To find these points, solve $\begin{cases} y=m(x+1) \\ x^2+y^2=1 \end{cases}$ for (x,y).

$x^2+\big(m(x+1)\big)^2=1$ (we have eliminated y from this equation)

This is a quadratic equation in x with rational coefficients.

Since $x=-1$ is one rational root, the other root must also be rational so (x,y) is rational. Every $m\in\mathbb{Q}$ gives a rational point on the unit circle.

Starting over, we give a completely algebraic approach to parameterizing the primitive Pythagorean triples.

$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ ring of integers.

$\mathbb{N} = \{1, 2, 3, 4, 5, \dots\}$ positive integers.

$\mathbb{N}$ has unique factorization. Every $n \in \mathbb{N}$ factors uniquely as a product of prime numbers $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, \dots$

i.e. if $n = p_1 p_2 \cdots p_k = q_1 q_2 \cdots q_\ell$ where all p_i, q_j are primes then $k = \ell$ and $p_i = q_i$ after re-indexing if necessary.

eg. $12 = 2 \times 6 = 2 \times 2 \times 3$ is a prime factorization of 12.

$12 = 3 \times 4 = 3 \times 2 \times 2$ " " " "

$1 = 1$ is a prime factorization with 0 prime factors.

A prime number is an integer $n > 1$ which is not of the form ab $(a, b \in \mathbb{N},\ a, b > 1)$.

We'll assume unique factorization for now but later, we'll have to explain this.

$\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$

x	x^2
0	0
1	1

$\gcd(a, b)$ = greatest common divisor of a, b

for $a, b \in \mathbb{N}$ eg. $\gcd(40, 68) = 2 \times 2 = 4$.

($40 = 2 \times 2 \times 2 \times 5$, $68 = 2 \times 2 \times 17$)

$(3, 4, 5)$, $(4, 3, 5)$

Pythagorean triple (a, b, c), a, b, c positive integers with $a^2 + b^2 = c^2$

(a, b, c) is primitive if it's not a scalar multiple (ka', kb', kc') with $k > 1$. $(6, 8, 10) = 2(3, 4, 5)$ is imprimitive.

If (a, b, c) is a primitive Pythagorean triple, what can we say about the parity of a, b, c?

parity — the quality of being even or odd

a, b, c can't all be even and they can't all be odd. In fact one must be even and the other two must be odd.

Can a, b be odd and c even? No.

Integers mod 4 $\mathbb{Z}/4\mathbb{Z} = \{0, 1, 2, 3\}$

x	x^2
0	0
1	1
2	0
3	1

If a, b are odd then $a^2 + b^2 \equiv 2 \bmod 4$ but if c is even then $c^2 \equiv 0 \bmod 4$

There is no Pythagorean triple (a, b, c) with a, b odd.

So every primitive Pythagorean triple is either (even, odd, odd) or (odd, even, odd).

Without loss of generality, take (odd, even, odd) ie. a, c odd, b even.

We will prove:

Theorem Every primitive Pythagorean triple has the form $(a,b,c) = (m^2-n^2,\ 2mn,\ m^2+n^2)$ for a unique pair of relatively prime integers m, n of opposite parity (i.e. one even, the other odd) with $m > n \geq 1$. (Or with a,b reversed.) Every such triple is a primitive Pythagorean triple.

Towards the proof, let's observe that in a primitive Pythagorean triple (a,b,c), any two of a,b,c are relatively prime i.e. $\gcd(a,b) = 1 = \gcd(a,c) = \gcd(b,c)$. Why?

Suppose (a,b,c) is not primitive, i.e. $(a,b,c) = (ka', kb', kc')$ with $k \geq 2$. Then $\gcd(a,b) \neq 1$ $(\gcd(a,b) \geq k)$
$\gcd(a,c) \neq 1$
$\gcd(b,c) \neq 1$.

Suppose (a,b,c) is a primitive Pythagorean triple. Why must $\gcd(a,b) = 1$?
Why must $\gcd(a,c) = 1$?
Why must $\gcd(b,c) = 1$?

Aside

Subtlety: The triple $(6,10,15)$ is primitive: it is not of the form $(6,10,15) = k(a,b,c)$, $k,a,b,c \in \mathbb{N}$, $k > 1$.
But $\gcd(6,10) = 2$, $\gcd(6,15) = 3$, $\gcd(10,15) = 5$. No two of $6,10,15$ are relatively prime.
Of course $(6,10,15)$ is not Pythagorean.

Given a primitive Pythagorean triple (a,b,c), $a^2+b^2 = c^2$ if $\gcd(a,b) > 1$ then there is a prime number p which is a factor of both a and b. But then p is a factor of a^2+b^2 so p is a factor of c^2 so p is a factor of c.
Then $a = pa'$, $b = pb'$, $c = pc'$, $(a,b,c) = p(a',b',c')$, $a', b', c' \in \mathbb{N}$. Then (a,b,c) is imprimitive.

What about $a^n + b^n = c^n$? $(a,b,c,n$ positive integers) For $n > 2$ there are no solutions.
This was known as Fermat's Last Theorem. Proved about 30 years ago by Andrew Wiles and others.

Given a primitive Pythagorean triple (a,b,c), $a^2+b^2=c^2$, we have $\gcd(a,b)=1$, $\gcd(a,c)=1$, $\gcd(b,c)=1$.
Without loss of generality, a,c are odd, b is even. Then $b^2 = c^2-a^2 = (c+a)(c-a)$ (even, even, even, even) so $\left(\frac{b}{2}\right)^2 = \frac{c+a}{2}\cdot\frac{c-a}{2}$.

$\frac{b}{2}\in\mathbb{N}$, $\frac{c+a}{2}\in\mathbb{N}$, $\frac{c-a}{2}\in\mathbb{N}$

Write $m=\frac{c+a}{2}$, $n=\frac{c-a}{2}$ so $m,n\in\mathbb{N}=\{1,2,3,\cdots\}$ positive integers.

$m>n\geq 1$. Then $\gcd(m,n)=1$. Why? If not then there is a prime p which is a factor of both m and n. Then $m+n=c$ is a multiple of p and
$m-n=a$ ---------- p. This is impossible since $\gcd(a,c)=1$.

$\left(\frac{b}{2}\right)^2 = m\cdot n$ An integer squared equals mn where m,n are relatively prime. Then m and n must both be squares. This fact follows directly from considering the prime factorization on both sides. We will discuss uniqueness of prime factorization later.

eg. $10^2=100=mn$

$100 = 100\times 1 = 50\times 2 = 25\times 4 = 20\times 5 = 10\times 10 = 5\times 20 = 4\times 25 = 2\times 50 = 1\times 100$

Aside

$m=M^2$, $n=N^2$, $M,N\in\mathbb{N}$.

$\left(\frac{b}{2}\right)^2 = M^2N^2$

$b^2 = 4M^2N^2$

$b = \pm 2MN$

$b = 2MN$

$c=m+n=M^2+N^2$

$a=m-n=M^2-N^2$

$M>N\geq 1$

$\gcd(M,N)=1$

If M,N are both odd then a,c would be even which is not true. So M,N must have opposite parity (one is even; the other is odd).

Are there infinitely many primes of the form n^2+1? e.g. $1^2+1=2$, $2^2+1=5$, $4^2+1=17$, etc. We believe the answer is "yes" but the problem is open.

Goldbach's Conjecture: Is every even number >2 a sum of two primes?
eg. $4=2+2$, $6=3+3$, $8=3+5$, $10=5+5=3+7$, $12=5+7$, $14=7+7=3+11$

The Riemann hypothesis: more about this later this semester. (Biggest open problem in mathematics)

There are infinitely many prime numbers $2,3,5,7,11,13,17,19,23,29,31,\dots$

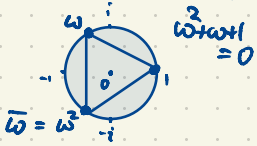
Are there infinitely many twin primes? e.g. $3,5$ $5,7$, $11,13$ $17,19$ $29,31$ etc.

Importance of Fermat's Last Theorem:
Try to use an idea similar to proof of classification of primitive Pythagorean triples.
eg. to show $x^3+y^3=z^3$ has no solution in positive integers $x,y,z\in\mathbb{N}$:

$$y^3=z^3-x^3=(z-x)(z^2+xz+x^2)=(z-x)(z-\omega x)(z-\omega^2 x)$$

Each of $z-x=a^3$, $z-\omega x=b^3$, $z-\omega^2 x=c^3$ $\quad a,b,c\in\mathbb{Z}[\omega]=\{r+s\omega : r,s\in\mathbb{Z}\}$ is the ring of Eisenstein integers.

$\omega=e^{2\pi i/3}=\frac{-1+\sqrt{3}i}{2}$, $\omega^2+\omega+1=0$



This leads to a contradiction, so we get a proof of Fermat's Last Theorem in the case of exponent 3.
This idea works a lot of the time so we can prove $x^n+y^n=z^n$ has no solution for certain values of n.
The argument fails for (most) many values of n because of the failure of unique factorization.
One early goal of our course: explain why $\mathbb{Z}$ has unique factorization and most similar rings do not have unique factorization.

Back to foundations of arithmetic of $\mathbb{Z}$. See handout on the integers on the course website.

$a, b, c, \ldots$ are integers: $a, b, c, \ldots \in \mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$

We say a <u>divides</u> b if $b = ka$ for some $k \in \mathbb{Z}$. (written $a \mid b$).

eg. 3 divides $6 = 3 \cdot 2$
3 divides $3 = 3 \cdot 1$
3 divides $-12 = -4 \cdot 3$
3 divides $0 = 0 \cdot 3$
3 does not divide 5.

$a \mid b \iff$ a divides b
$\iff$ b is a multiple of a
$\iff$ a is a <u>divisor</u> of b
$\iff$ a is a "<u>factor</u>" of b.

The divisors of 12 are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$. There are exactly twelve numbers that divide 12: i.e. $-12, -6, -4, -3, -2, -1, 1, 2, 3, 4, 6, 12$.

The divisors of 10 are $\pm 1, \pm 2, \pm 5, \pm 10$. (There are eight divisors of 10).

The divisors of -14 are $\pm 1, \pm 2, \pm 7, \pm 14$.

The divisors of 5 are $\pm 1, \pm 5$.

The divisors of 1 are ± 1. (two divisors)

The divisors of 0 are $0, \pm 1, \pm 2, \pm 3, \pm 4, \cdots$

$0 = 17 \cdot 0$

Given two integers a, b, their common divisors:

The divisors of 68 are $\pm 1, \pm 2, \pm 4, \pm 17, \pm 34, \pm 68$
The divisors of 10 are $\pm 1, \pm 2, \pm 5, \pm 10$.
The common divisors of 68 and 10 are $\pm 1, \pm 2$ i.e. $-2, -1, 1, 2$.
The greatest common divisor of 68 and 10 is 2.

Range of difficulty of computational problems

add, subtract, multiply:	easy (with modest computational tools)
factorization	hard
find gcd	easy
testing primality	easy

Compute $\gcd(a, b)$ efficiently using Euclid's Algorithm

$\gcd(68, 0) = 68$
$\gcd(0, 0)$ is undefined

Divisors of 68: $\pm 1, \pm 2, \pm 4, \pm 17, \pm 34, \pm 68$
Divisors of 0: $0, \pm 1, \pm 2, \pm 3, \pm 4, \cdots$
Common divisors of 68 and 0: $\pm 1, \pm 2, \pm 4, \pm 17, \pm 34, \pm 68$
Greatest ... : 68

Computing $\gcd(513, 381) = 3$

$513 = 1 \times 381 + 132$
$381 = 2 \times 132 + 117$
$132 = 1 \times 117 + 15$
$117 = 7 \times 15 + 12$
$15 = 1 \times 12 + 3$
$12 = 4 \times 3 + 0$

The $\gcd(a,b)$ is the last nonzero remainder

Division Algorithm: Given $a, d \in \mathbb{Z}$ with $d > 0$, there exist unique $q, r \in \mathbb{Z}$ such that $a = qd + r$, $0 \le r < d$.
(If $r = 0$ we say d divides a, i.e. $d \mid a$.)
r is the remainder; q is the quotient.

$(a, b \in \mathbb{Z}$, not both zero)

the extended form of Euclid's Algorithm: $3 = 513r + 381s$, $r, s \in \mathbb{Z}$.

$\gcd(513, 381)$ as an integer linear combination of a and b.

We can write $\gcd(a,b)$ as an integer linear combination of a and b.

$3 = 513 \times 26 + 381 \times (-35)$

This tells us: $\{513r + 381s : r, s \in \mathbb{Z}\} = \{3t : t \in \mathbb{Z}\} = \{\ldots, -9, -6, -3, 0, 3, 6, 9, 12, \ldots\}$