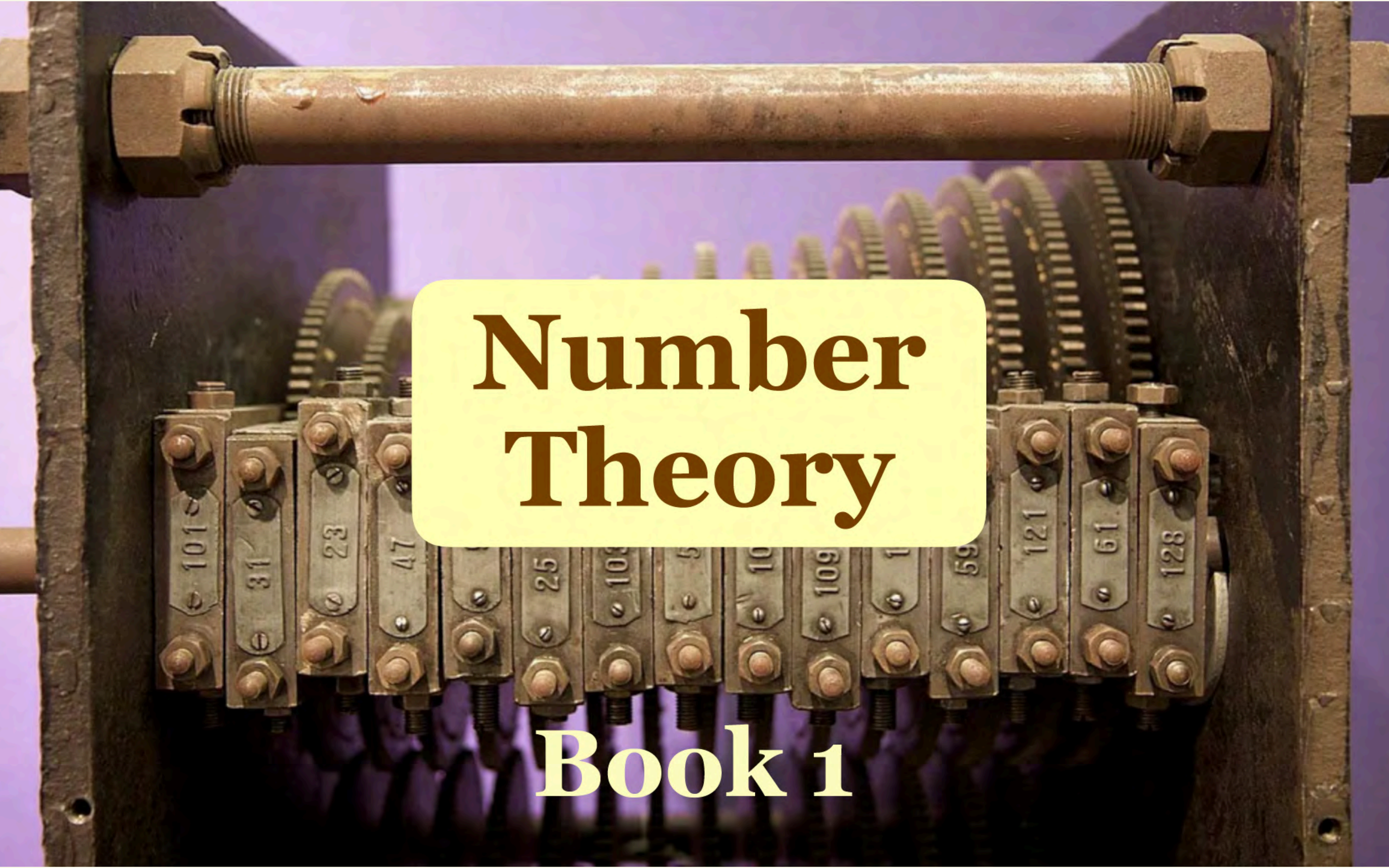
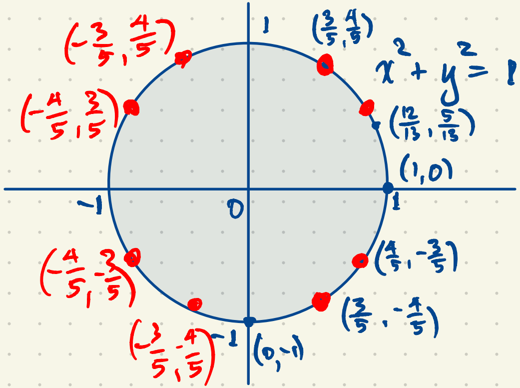
Number Theory

Book 1





How many points on the circle $x^2+y^2=1$ $(x,y \in \mathbb{Q})$ have rational number coordinates?

Not $(\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$

Are there infinitely many "rational points" on the unit circle?

$(\frac{3}{5}, \frac{4}{5}) \longleftrightarrow 3^2+4^2=5^2$ solution of $x^2+y^2=z^2$ $(x,y,z \in \mathbb{Z})$

A Pythagorean triple is a triple (a,b,c) of positive integers a,b,c, satisfying $a^2+b^2=c^2$.

eg. $(3,4,5)$, $(6,8,10)$, $(9,12,15)$, $(5,12,13)$, ...

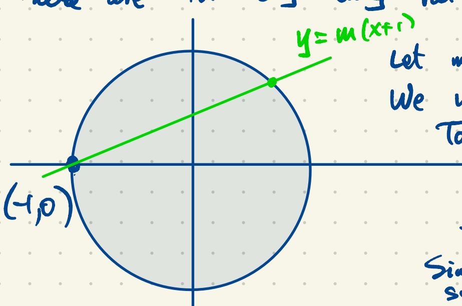
A triple (a,b,c) is primitive if it is not an integer scalar multiple of a smaller triple eg. $(3,4,5)$ is primitive; $(6,8,10) = 2(3,4,5)$ is imprimitive, as is $(9,12,15) = 3(3,4,5)$.

There are infinitely many primitive Pythagorean triples.

The triple $(3,4,5)$ yields eight rational points $(\pm\frac{3}{5}, \pm\frac{4}{5})$, $(\pm\frac{4}{5}, \pm\frac{3}{5})$. So does $(4,3,5)$

Theorem There are infinitely many rational points on the unit circle $x^2+y^2=1$. See Chapter 3.

Proof



Let $m \in \mathbb{Q}$. Consider the line $y = m(x+1)$ through $(-1,0)$.
We will see that this line intersects the circle in two rational points.
To find these points, solve $\begin{cases} y = m(x+1) \\ x^2+y^2=1 \end{cases}$ for (x,y).

$$x^2 + \left(m(x+1)\right)^2 = 1$$ (we have eliminated y from this equation)

This is a quadratic equation in x with rational coefficients.
Since $x=-1$ is one rational root, the other root must also be rational
so (x,y) is rational. Every $m \in \mathbb{Q}$ gives a rational point on the unit circle

Starting over, we give a completely algebraic approach to parameterizing the primitive Pythagorean triples.

$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ ring of integers.

$\mathbb{N} = \{1, 2, 3, 4, 5, \dots\}$ positive integers.

$\mathbb{N}$ has unique factorization. Every $n \in \mathbb{N}$ factors uniquely as a product of prime numbers 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, ...

i.e. if $n = p_1 p_2 \cdots p_k = q_1 q_2 \cdots q_l$ where all p_i, q_j are primes then $k = l$ and $p_i = q_i$ after re-indexing if necessary.

eg. $12 = 2 \times 6 = 2 \times 2 \times 3$ is a prime factorization of 12.

$12 = 3 \times 4 = 3 \times 2 \times 2$ " " "

$1 = 1$ is a prime factorization with 0 prime factors.

A prime number is an integer $n > 1$ which is not of the form ab $(a, b \in \mathbb{N},\ a, b > 1)$.

We'll assume unique factorization for now but later, we'll have to explain this.

$\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$

x	x^2
0	0
1	1

$\gcd(a, b)$ = greatest common divisor of a, b

for $a, b \in \mathbb{N}$

eg. $\gcd(\underset{2\times2\times2\times5}{40}, \underset{2\times2\times17}{68}) = 2 \times 2 = 4$.

$(3, 4, 5)$, $(4, 3, 5)$

Pythagorean triple (a, b, c), a, b, c positive integers with $a^2 + b^2 = c^2$

(a, b, c) is primitive if it's <u>not</u> a scalar multiple (ka', kb', kc') with $k > 1$. $(6, 8, 10) = 2(3, 4, 5)$ is imprimitive.

If (a, b, c) is a primitive Pythagorean triple, what can we say about the <u>parity</u> of a, b, c? ← the quality of being even or odd

a, b, c can't all be even and they can't all be odd. In fact one must be even and the other two must be odd.

Can a, b be odd and c even? No.

Integers mod 4 $\mathbb{Z}/4\mathbb{Z} = \{0, 1, 2, 3\}$

x	x^2
0	0
1	1
2	0
3	1

If a, b are odd then $a^2 + b^2 \equiv 2 \bmod 4$ <u>but</u> if c is even then $c^2 \equiv 0 \bmod 4$

There is no Pythagorean triple (a, b, c) with a, b odd.

So every primitive Pythagorean triple is either (even, odd, odd) or (odd, even, odd).

Without loss of generality, take (odd, even, odd) ie. a, c odd, b even.

We will prove:

Theorem Every primitive Pythagorean triple has the form $(a,b,c) = (m^2-n^2,\ 2mn,\ m^2+n^2)$ for a unique pair of relatively prime integers m, n of opposite parity (i.e. one even, the other odd) with $m > n \geq 1$. (Or with a,b reversed)
Every such triple is a primitive Pythagorean triple.

Towards the proof, let's observe that in a primitive Pythagorean triple (a,b,c), any two of a,b,c are relatively prime i.e. $\gcd(a,b)=1=\gcd(a,c)=\gcd(b,c)$. Why?

Suppose (a,b,c) is not primitive, i.e. $(a,b,c)=(ka', kb', kc')$ with $k \geq 2$. Then $\gcd(a,b) \neq 1$ $(\gcd(a,b) \geq k)$
$\gcd(a,c) \neq 1$
$\gcd(b,c) \neq 1$.

Suppose (a,b,c) is a primitive Pythagorean triple. Why must $\gcd(a,b)=1$?
Why must $\gcd(a,c)=1$?
Why must $\gcd(b,c)=1$?

Aside

Subtlety: The triple $(6,10,15)$ is primitive: it is not of the form $(6,10,15)=k(a,b,c)$, $k,a,b,c \in \mathbb{N}$, $k>1$.
But $\gcd(6,10)=2$, $\gcd(6,15)=3$, $\gcd(10,15)=5$. No two of $6,10,15$ are relatively prime.
Of course $(6,10,15)$ is not Pythagorean.

Given a primitive Pythagorean triple (a,b,c), $a^2+b^2=c^2$, if $\gcd(a,b)>1$ then there is a prime number p which is a factor of both a and b. But then p is a factor of a^2+b^2 so p is a factor of c^2 so p is a factor of c.
Then $a=pa'$, $b=pb'$, $c=pc'$, $(a,b,c)=p(a',b',c')$, $a',b',c' \in \mathbb{N}$. Then (a,b,c) is imprimitive.

What about $a^n+b^n=c^n$? (a,b,c,n positive integers) For $n>2$ there are no solutions.
This was known as Fermat's Last Theorem. Proved about 30+ years ago by Andrew Wiles and others.

Given a primitive Pythagorean triple (a,b,c), $a^2+b^2=c^2$, we have $\gcd(a,b)=1$, $\gcd(a,c)=1$, $\gcd(b,c)=1$.
Without loss of generality, a,c are odd, b is even. Then $\underbrace{b^2}_{\text{even}} = \underbrace{c^2-a^2}_{\text{even}} = \underbrace{(c+a)}_{\text{even}}\underbrace{(c-a)}_{\text{even}}$ so $\left(\frac{b}{2}\right)^2 = \frac{c+a}{2}\cdot\frac{c-a}{2}$.

$\frac{b}{2}\in\mathbb{N}$ $\quad \frac{c+a}{2}\in\mathbb{N}$ $\quad \frac{c-a}{2}\in\mathbb{N}$

Write $m=\frac{c+a}{2}$, $n=\frac{c-a}{2}$ so $m,n\in\mathbb{N}=\{1,2,3,\dots\}$ positive integers.

$m>n\geq 1$. Then $\gcd(m,n)=1$. Why? If not then there is a prime p which is a factor of both m and n. Then $m+n=c$ is a multiple of p and
$m-n=a$ — — — — — — p. This is impossible since $\gcd(a,c)=1$.

$\left(\frac{b}{2}\right)^2 = m\cdot n$ An integer squared equals mn where m,n are relatively prime. Then m and n must both be squares. This fact follows directly from considering the prime factorization on both sides. We will discuss uniqueness of prime factorization later.

eg. $10^2=100=mn$

$100 = 100\times 1$
$= ~~50\times 2~~$
$= 25\times 4$
$= ~~20\times 5~~$
$= ~~10\times 10~~$
$= ~~5\times 20~~$
$= 4\times 25$
$= ~~2\times 50~~$
$= 1\times 100$

Aside

$m=M^2$, $n=N^2$, $M,N\in\mathbb{N}$.

$$\left(\frac{b}{2}\right)^2 = M^2N^2$$
$$b^2 = 4M^2N^2$$
$$b = \pm 2MN$$
$$b = 2MN$$

$c=m+n=M^2+N^2$
$a=m-n=M^2-N^2$

$M>N\geq 1$
$\gcd(M,N)=1$

If M,N are both odd then a,c would be even which is not true. So M,N must have opposite parity (one is even; the other is odd).

Are there infinitely many primes of the form n^2+1? e.g. $1^2+1=2$, $2^2+1=5$, $4^2+1=17$ etc.

We believe the answer is "yes" but the problem is open.

Goldbach's Conjecture: Is every even number >2 a sum of two primes?
eg. $4=2+2$, $6=3+3$, $8=3+5$, $10=5+5=3+7$, $12=5+7$, $14=7+7=3+11$

The Riemann Hypothesis: more about this later this semester. (Biggest open problem in mathematics)

There are infinitely many prime numbers $2,3,5,7,11,13,17,19,23,29,31,\ldots$

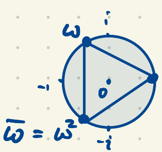
Are there infinitely many twin primes? e.g. $3,5$ $5,7$, $11,13$ $17,19$ $29,31$ etc.

Importance of Fermat's Last Theorem:

Try to use an idea similar to proof of classification of primitive Pythagorean triples.
eg. to show $x^3+y^3=z^3$ has no solution in positive integers $x,y,z\in\mathbb{N}$:

$$y^3 = z^3-x^3 = (z-x)(z^2+xz+x^2) = (z-x)(z-\omega x)(z-\omega^2 x)$$

$\omega = e^{2\pi i/3} = \frac{-1+\sqrt{-3}}{2}$



$\omega^2+\omega+1=0$

Each of
$z-x=a^3$
$z-\omega x=b^3$
$z-\omega^2 x=c^3$

$a,b,c\in\mathbb{Z}[\omega]=\{r+s\omega : r,s\in\mathbb{Z}\}$ is the ring of Eisenstein integers.

This leads to a contradiction, so we get a proof of Fermat's Last Theorem in the case of exponent 3.
This idea works a lot of the time so we can prove $x^n+y^n=z^n$ has no solution for certain values of n.
The argument fails for many (most) values of n because of the failure of unique factorization.
One early goal of our course: explain why $\mathbb{Z}$ has unique factorization and most similar rings do not have unique factorization.

Back to foundations of arithmetic of $\mathbb{Z}$. See handout on the integers on the course website.

$a, b, c, \dots$ are integers: $a, b, c, \dots \in \mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$

We say a <u>divides</u> b if $b = ka$ for some $k \in \mathbb{Z}$. (written $a \mid b$).

$a \mid b \iff$ a divides b
$\iff$ b is a multiple of a
$\iff$ a is a <u>divisor</u> of b
$\iff$ a is a "<u>factor</u>" of b.

eg. 3 divides $6 = 3 \cdot 2$
3 divides $3 = 3 \cdot 1$
3 divides $-12 = -4 \cdot 3$
3 divides $0 = 0 \cdot 3$
3 does not divide 5.

The divisors of 12 are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$. There are exactly twelve numbers that divide 12: i.e. $-12, -6, -4, -3, -2, -1, 1, 2, 3, 4, 6, 12$.

The divisors of 10 are $\pm 1, \pm 2, \pm 5, \pm 10$. (There are eight divisors of 10).

The divisors of -14 are $\pm 1, \pm 2, \pm 7, \pm 14$.

The divisors of 5 are $\pm 1, \pm 5$.

The divisors of 1 are ± 1. (two divisors)

The divisors of 0 are $0, \pm 1, \pm 2, \pm 3, \pm 4, \dots$

$0 = 17 \cdot 0$

Given two integers a, b, their common divisors:

The divisors of 68 are $\pm 1, \pm 2, \pm 4, \pm 17, \pm 34, \pm 68$
The divisors of 10 are $\pm 1, \pm 2, \pm 5, \pm 10$.
The common divisors of 68 and 10 are $\pm 1, \pm 2$ i.e. $-2, -1, 1, 2$.
The greatest common divisor of 68 and 10 is 2.

Range of difficulty of computational problems

add, subtract, multiply:	easy	(with modest computational tools)
factorization	hard	
find gcd	easy	
testing primality	easy	

Compute $\gcd(a,b)$ efficiently using Euclid's Algorithm

$\gcd(68, 0) = 68$

$\gcd(0,0)$ is undefined

Divisors of 68: $\pm 1, \pm 2, \pm 4, \pm 17, \pm 34, \pm 68$
Divisors of 0: $0, \pm 1, \pm 2, \pm 3, \pm 4, \dots$
Common divisors of 68 and 0: $\pm 1, \pm 2, \pm 4, \pm 17, \pm 34, \pm 68$
Greates - - - - - - - - - : 68

Computing gcd(513, 381) = 3

$513 = 1\times 381 + 132$
$381 = 2\times 132 + 117$
$132 = 1\times 117 + 15$
$117 = 7\times 15 + 12$
$15 = 1\times 12 + 3$
$12 = 4\times 3 + 0$

The gcd(a,b) is the last nonzero remainder

Division Algorithm: Given $a, d \in \mathbb{Z}$ with $d > 0$, there exist unique $q, r \in \mathbb{Z}$ such that $a = qd + r$, $0 \le r < d$.
(If $r = 0$ we say d divides a, i.e. $d \mid a$.)
r is the remainder; q is the quotient.

($a, b \in \mathbb{Z}$, not both zero)

the extended form of Euclid's Algorithm: $3 = 513r + 381s$, $r, s \in \mathbb{Z}$.

gcd(513, 381) as an integer linear combination of a and b.

We can write gcd(a,b) as an integer linear combination of a and b.

$3 = 513 \times \underline{26} + 381 \times \underline{(-35)}$

This tells us: $\{513r + 381s : r, s \in \mathbb{Z}\} = \{3t : t \in \mathbb{Z}\} = \{\dots, -9, -6, -3, 0, 3, 6, 9, 12, \dots\}$

Shorthand:

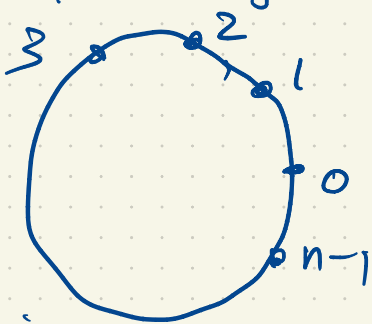
513	381	
1	0	513
0	1	381
1	−1	132
−2	3	117
3	−4	15
−23	31	12
26	−35	$\boxed{3}$
*	*	0

i.e. $1\times 513 + 0\times 381 = 513$
i.e. $1\times 513 - 1\times 381 = 132$
i.e. $-2\times 513 + 3\times 381 = 117$

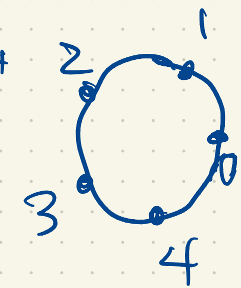
i.e. $\gcd(513, 381) = 3 = 26\times 513 - 35\times 381$

Congruences; modular arithmetic

If n is a positive integer then we write $a \equiv b \mod n$ whenever $a-b$ is divisible by n
$\iff n \mid a-b$
$\iff a-b$ is a multiple of n



eg. integers mod 5: Every $a \in \mathbb{Z}$ is congruent mod 5 to exactly one of 0, 1, 2, 3 or 4



$7 \times 8 = 56 \equiv 1 \mod 5$ (56 is congruent to 1 mod 5 because $56-1=55$ is divisible by 5).

$33 \equiv 78 \mod 5$ because $33-78$ is divisible by 5.

$33 \equiv -7 \mod 5$

$33 \not\equiv 6$ because $33-6=27$ is not divisible by 5.

Look in early chapters of textbook and my handouts including integers

If (a,b,c) is a primitive Pythagorean triple then $c \equiv 1 \mod 4$.
($c = m^2+n^2$ where m,n are integers of opposite parity so $c \equiv 0^2+1^2 \equiv 1 \mod 4$)

Solve for $x, y \in \mathbb{Z}$: $\quad 3x + 5y \equiv 1 \mod 7$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad 4x + y \equiv 3 \mod 7$

x	y		
3	5	1	i.e. $3x+5y\equiv 1 \mod 7$
4	1	3	
0	6	4	i.e. $6y \equiv 4$
0	1	3	
3	0	-14	i.e. $3x \equiv -14 \equiv 0$
3	0	0	
1	0	0	

Solution: $x \equiv 0,\ y \equiv 3 \mod 7$.

Check: $3\cdot 0 + 5\cdot 3 \equiv 1 \mod 7$
$\quad\quad\quad 4\cdot 0 + 3 \equiv 3 \mod 7$.

i.e. $(x,y) \in \{(0,3), (7,3), (-7,-4), (-7,10), \dots\}$
infinitely many solutions in $\mathbb{Z}^2$.
(satisfying our congruences)

$$\mathbb{Z}/7\mathbb{Z} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}\} = \{0,1,2,\dots,6\}$$ abbreviated for simplicity

$\bar{0} = [0] = \{\dots, -14, -7, 0, 7, 14, 21, \dots\}$
$\bar{1} = [1] = \{\dots, -13, -6, 1, 8, 15, 22, \dots\}$
etc.

$5 \times 6 = 2$ in $\mathbb{Z}/7\mathbb{Z}$
i.e. $\bar{5} \times \bar{6} = \bar{2}$
$5 \times 6 \equiv 2 \mod 7$ in $\mathbb{Z}$

Solve: $\begin{matrix} 3x + 5y = 1 \\ 4x + y = 3 \end{matrix}$ for $x, y \in \mathbb{Z}/7\mathbb{Z} = \mathbb{F}_7 = \{0,1,2,\dots,6\}$ finite field of order 7

Answer: $(x,y) = (0,3)$ is the unique solution in $\mathbb{F}_7^{\,2} = \{(x,y) : x, y \in \mathbb{F}_7\}$.

Equivalently: Solve $\begin{cases} 3x + 5y \equiv 1 \mod 7 \\ 4x + y \equiv 3 \mod 7 \end{cases}$ for $x, y \in \mathbb{Z}$.

There are infinitely many solutions of this system of two linear congruences in two unknowns $x, y \in \mathbb{Z}$, namely $x \equiv 0 \mod 7$, $y \equiv 3 \mod 7$. i.e. $x \in \{\dots, -14, -7, 0, 7, 14, 21, \dots\}$, $y \in \{\dots, -11, -4, 3, 10, 17, 24, \dots\}$

Solve $4x = 5$ for $x \in \mathbb{F}_7 = \mathbb{Z}/7\mathbb{Z} = \{0,1,2,3,4,5,6\}$ (integers mod 7).

If we were working in $\mathbb{R}$ or in $\mathbb{Q}$, $x = \frac{5}{4}$. In $\mathbb{F}_7$, the answer $x = \frac{5}{4}$ is technically correct but this is not simplified.

x	0	1	2	3	4	5	6
$4x$	0	4	1	5	2	6	3

The unique solution is $x = \frac{5}{4} = 3$.

$$\frac{5}{4} = \frac{12}{4} = \frac{6}{2} = 3$$

In $\mathbb{Z}$, solve the congruence $4x \equiv 5 \mod 7$. Solution: $x \equiv 3 \mod 7$ i.e. $x \in \{\ldots, -11, -4, 3, 10, 17, \ldots\}$

If x is a positive integer such that the last two decimal digits of $41x$ are 01 i.e. $41x = \square\square\square\square\square\square 01$, what are the last two digits of x?

Solve $41x \equiv 1 \mod 100$. We need the inverse of 41 in $\mathbb{Z}/100\mathbb{Z} = \{0, 1, 2, \ldots, 99\}$

$\gcd(100, 41) = 1 = (16) \times 100 + (-39) \times 41 = 1600 - 1599$

check:

100	41	
1	0	100
0	1	41
1	-2	18
-2	5	5
7	-17	3
-9	22	2
16	-39	$\boxed{1}$
*	*	0

$1 \equiv 41 \times (-39) \equiv 41 \times 61$

Solution: $x \equiv 61 \mod 100$

i.e. the last two decimal digits of x are 61.

Note: $45x \equiv 1 \mod 100$ has no solution for $x \in \mathbb{Z}$. $\gcd(45, 100) = 5 \neq 1$.

$45x \equiv 5 \mod 100$ has multiple solutions mod 100.

Important fact: $\mathbb{Z}/7\mathbb{Z} = \mathbb{F}_7$ is a field: every nonzero element has an inverse.

This is because 7 is a prime number.

If p is prime then every one of $1, 2, \cdots, p-1$ has an inverse mod p.

$\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} = \{0, 1, 2, \cdots, p-1\}$ is a field. Every nonzero element has an inverse.

Use Euclid's algorithm to find inverses.

$\mathbb{Z}/100\mathbb{Z}$ is not a field. It's only a ring.

Let x be a positive integer such that the last two decimal digits of x^{67} are ...83. What are the last two digits of x? Hint: The last two digits of x^{23} are ... 03.

Another hint: 67 and 23 are relatively prime.

67	23	
1	0	67
0	1	23
1	−2	21
−1	3	2
11	−32	$\boxed{1}$
*	*	0

$\boxed{1} = \gcd(67, 23) = 11\times 67 - 32\times 23$

$x^{67} \equiv 83 \mod 100$

$x^{23} \equiv 3 \mod 100$

All congruences mod 100.

$$x^1 = x^{11\times 67 - 32\times 23} \equiv (x^{67})^{11} (x^{23})^{-32} \equiv 83^{11}\cdot 3^{-32}$$

$$\equiv 67 \cdot (3^{32})^{-1} \equiv 67 \cdot 41^{-1} \mod 100$$

$$\equiv 67 \times 61 \equiv 4087 \equiv 87$$

see previous page

Check: $87^{67} \equiv 83 \mod 100.$

$87^{23} \equiv 3 \mod 100.$

Write 61 as a sum of two squares. $61 = 5^2+6^2 = 25+36$
" 97 " " " " " . $97 = 9^2+4^2 = 81+16$

In each of these two cases, the solution is essentially unique.

$61 = 5^2+6^2 = 6^2+5^2 = (-5)^2+6^2 = (-5)^2+(-6)^2 = 6^2+(-5)^2 = \cdots$

there are exactly $\underline{\ 8\ }$ pairs of integers $(x,y)\in\mathbb{Z}^2$ satisfying $x^2+y^2=61$.

Solutions: $(5,6)$, $(6,5)$, $(-5,6)$, $(-6,5)$, $(-6,-5)$, $(6,-5)$, $(5,-6)$, $(-5,-6)$

Given a positive integer n, we may ask:
Can n be written as a sum of two squares?
If so, in how many ways? or how many essentially distinct ways?

1003 is not a sum of two squares. Every square $a^2 \equiv 0$ or $1 \bmod 4$.
So every sum of two squares is congruent to 0, 1 or 2 mod 4.
Since $1003 \equiv 3 \bmod 4$, it's not a sum of two squares.

The converse ("Every positive integer congruent to 0, 1 or 2 mod 4 is a sum of two squares") is false. (6 is the smallest counterexample.)

5917 is a sum of two squares since $5917 = 61\times 97 = (5^2+6^2)(9^2+4^2) =$