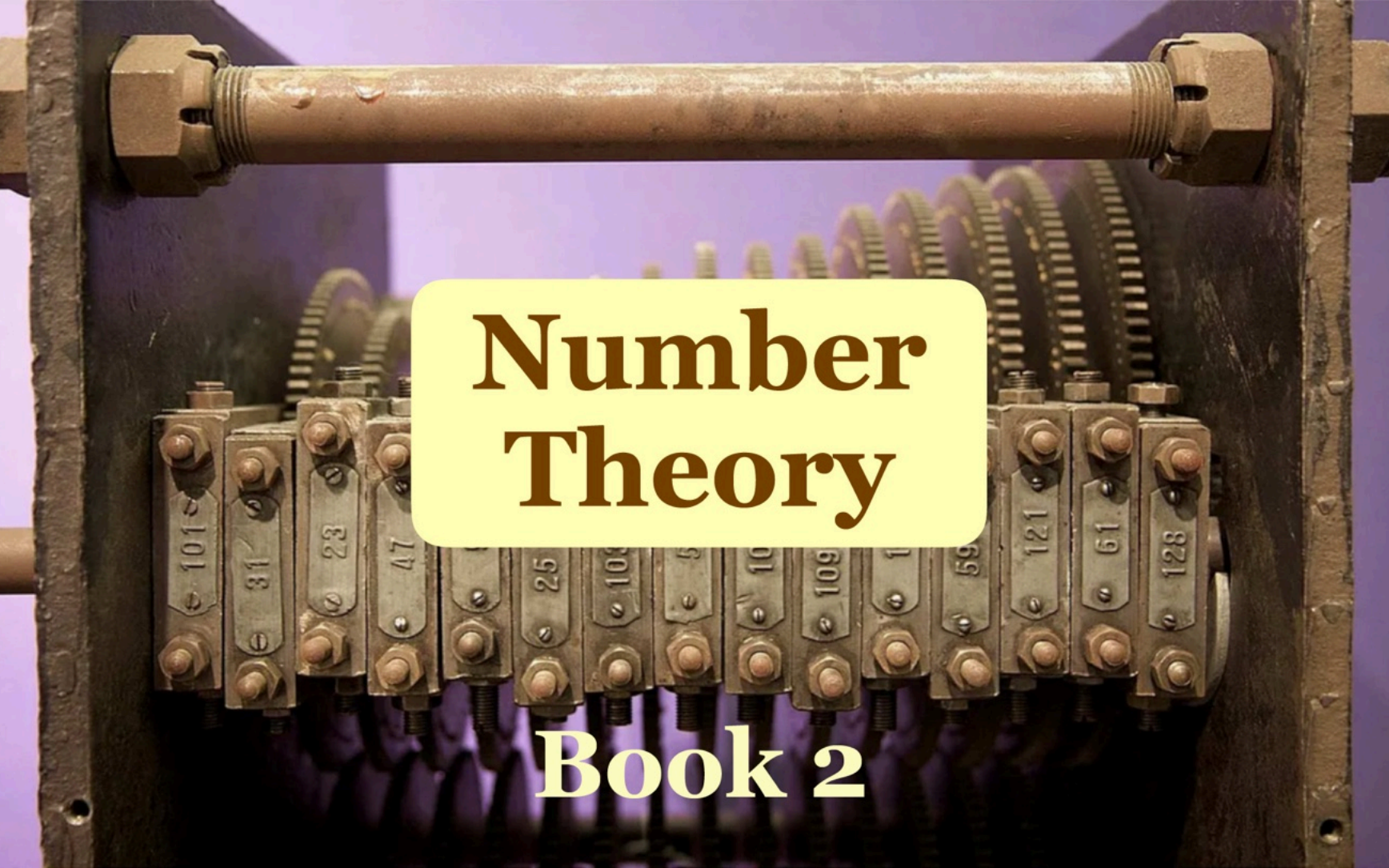
Number
Theory
Book 2

"Trick" (or technique) for identifying which small integers have the form x^2+y^2 or x^2+3y^2 or ...

Define $\theta(t) = \sum_{n=-\infty}^{\infty} t^{n^2} = 1 + 2t + 2t^4 + 2t^9 + 2t^{16} + 2t^{25} + \cdots = \sum_{n\in\mathbb{Z}} t^{n^2}$

$\theta(t)^2 = (1+2t+2t^4+2t^9+2t^{16}+2t^{25}+\cdots)(1+2t+2t^4+2t^9+2t^{16}+2t^{25}+\cdots) = 1 + 4t + 4t^2 + 4t^4 + 8t^5 + 4t^8 + \cdots + 12t^{25} + \cdots$

The powers of t appearing in this expansion are precisely the exponents expressible as a sum of two squares.
The coefficient of t^n on the right hand side is the number of solutions of $n = x^2+y^2$ $(x,y\in\mathbb{Z})$

eg. $25 = x^2+y^2$ has 12 solutions $(\pm5, 0)$, $(0,\pm5)$, $(\pm3, \pm4)$, $(\pm4, \pm3)$
$5 = x^2+y^2$ has 8 solutions $(\pm2, \pm1)$, $(\pm1, \pm2)$
$6 = x^2+y^2$ has 0 solutions.

$\theta(t)^3 = 1 + 6t + 12t^2 + 8t^3 + 6t^4 + 24t^5 + 24t^6 + 12t^8 + \cdots$

$1 = x^2+y^2+z^2$ has six solutions $(\pm1, 0, 0)$, $(0,\pm1, 0)$, $(0, 0, \pm1)$
$2 =$ ... twelve $(\pm1, \pm1, 0)$, $(\pm1, 0, \pm1)$, $(0, \pm1, \pm1)$
$7 = x^2+y^2+z^2$ has 0 solutions

$\theta(t)^4$ has positive coefficient of t^n for every positive integer n
Lagrange's Theorem: every positive integer is a sum of four squares

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Fundamental Theorem of Arithmetic: Every positive integer is uniquely expressible as a product of primes.
We'll explain exactly what this says and we'll prove it.

Fundamental Theorem of Calculus
" " of Linear Algebra
" " " Algebra

FTA = Fundamental Theorem of Arithmetic:
Positive integers factor uniquely as products of primes.

eg. $12 = 6\times 2 = 2\times 3\times 2$

$12 = 3\times 4 = 3\times 2\times 2$

$12 = (-2)\times(-6) = 2\times 6$

$12 = \boxed{1}\times 12$

Ignore factors of ± 1 in factorization: these are <u>units</u>. (invertible elements). In $\mathbb{Z}$, the only units are ± 1.

eg. $\mathbb{Z}[i] = \{a+bi : a,b\in\mathbb{Z}\}$ (commutative ring with identity 1) we again have unique factorization. The units are $\pm 1, \pm i$.

$i\cdot(-i) = 1$

$1\cdot 1 = 1$

$(-1)(-1) = 1$

$\mathbb{Z}[i]$ = "Gaussian integers"

Elements of R:
- zero
- units
- irreducible
- reducible.

$12 = 2\times 2\times 3 = \underbrace{(1+i)(1-i)(1+i)(1-i)3}_{\text{cannot be factored any further; they are irreducible.}} = (i-1)(-i-1)(1+i)(1-i)3$

$2+i = i(1-2i)$

$10 = 2\times 5 = (1+i)(1-i)(1+2i)(1-2i) = (1+i)(1-i)(2+i)(2-i)$ "migration of units"

$\times i$, $\times(-i)$

A (commutative ring with identity) has unique factorization if every nonzero element can be factored as a product of irreducible elements and this factorization is essentially unique (i.e. unique up to permutation of factors and migration of units.

Why don't we just say primes and composites instead of irreducible and reducible elements?

In $\mathbb{Z}$:
- zero: 0
- units: ± 1
- irreducible: $\pm 2, \pm 3, \pm 5, \pm 7, \pm 11, \cdots$
- reducible: $\pm 4, \pm 6, \pm 8, \pm 9, \pm 10, \pm 12, \cdots$

In $\mathbb{Z}[i]$:
- zero: 0
- units: $\pm 1, \pm i$
- irreducibles: $\pm 1 \pm i = \{1+i, 1-i, -1+i, -1-i\}$ (actually $3, -3, 3i, -3i$)
  $\pm 1 \pm 2i, \pm 2 \pm i$ etc.
- reducibles: 2 (actually $2, -2, 2i, -2i$)

A nonzero element $\alpha \in R$ is reducible if $\alpha \neq 0$, $\alpha \neq$ unit and $\alpha = \alpha_1 \alpha_2$ α_1, α_2 not units.

α is irreducible if the only factorizations $\alpha = \alpha_1 \alpha_2$ have either α_1 or α_2 is a unit.

Eg. in $\mathbb{Z}[i]$, 3 is irreducible. If $3 = \beta\gamma$, $\beta, \gamma \in \mathbb{Z}[i]$ then either $\beta \in \{\pm 1, \pm i\}$ or $\gamma \in \{\pm 1, \pm i\}$

2 is reducible since $2 = (1+i)(1-i)$, neither $1+i$ nor $1-i$ is a unit.

Eg. the ring $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\}$ does not have unique factorization.

$\mathbb{Z}[\sqrt{6}] = \{a + b\sqrt{6} : a, b \in \mathbb{Z}\}$ . . . . . .

An example of non-unique factorization in $\mathbb{Z}[\sqrt{6}]$: $6 = 2 \times 3 = \sqrt{6} \times \sqrt{6}$

$2, 3, \sqrt{6}$ are irreducible in $\mathbb{Z}[\sqrt{6}]$.

But be careful: $\mathbb{Z}[\sqrt{6}]$ has infinitely many units.

Pell's equation $x^2 - 6y^2 = 1$ has solutions $(\pm 1, 0)$, $(\pm 5, \pm 2)$

$(5 + 2\sqrt{6})(5 - 2\sqrt{6}) = 1$ in $\mathbb{Z}[\sqrt{6}]$

$x^2 - 6y^2 = -1$ has no integer solutions.

(Look at the equation mod 3)

In $\mathbb{Z}[\sqrt{-5}]$, the only units are ± 1.

Unique factorization is not universal.

Difference between $\mathbb{Z}[i], \mathbb{Z}, \mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}[\sqrt{-5}], \mathbb{Z}[\sqrt{6}]$

$\mathbb{Z}[i] = \mathbb{Z}[\sqrt{-1}]$

unique factorization

$15 = 3 \times 5 = (-3) \times (-5)$ in $\mathbb{Z}$

$15 = 3 \times (2+i)(2-i)$ in $\mathbb{Z}[i]$

$= 3 \times (1+2i)(1-2i)$

$(2-i)i = 1+2i$; $(2+i)(-i) = 1-2i$

migration of units

$15 = 3 \times 5 = (3+3\sqrt{2})(-5+5\sqrt{2})$ in $\mathbb{Z}[\sqrt{2}]$

$\times (1+\sqrt{2})$ $\quad \times (-1+\sqrt{2})$

$(1+\sqrt{2})(-1+\sqrt{2}) = 1$

Infinitely many units in $\mathbb{Z}[\sqrt{2}]$: solutions of $(a+b\sqrt{2})(a-b\sqrt{2}) = 1$

$a^2 - 2b^2 = 1$

$\pm(1+\sqrt{2})^n = \pm 1, \pm 1 \pm \sqrt{2}, \pm 3 \pm 2\sqrt{2}, \cdots$

$3^2 - 2 \cdot 2^2 = (3+2\sqrt{2})(3-2\sqrt{2}) = 1$

$6 = 2 \times 3 = (1+\sqrt{-5})(1-\sqrt{-5})$

where all factors $2, 3, 1+\sqrt{-5}, 1-\sqrt{-5}$ are irreducible in $\mathbb{Z}[\sqrt{-5}]$

It holds in $\mathbb{Z}, \mathbb{Z}[i], \mathbb{Z}[\sqrt{2}], \mathbb{Z}[\sqrt{3}], \mathbb{Z}[\sqrt{5}], \mathbb{Z}[\sqrt{-2}], \mathbb{Z}[\sqrt{-3}]$

Not in $\mathbb{Z}[\sqrt{-5}]$ or $\mathbb{Z}[\sqrt{6}]$.

non-unique factorization

In $\mathbb{Z}[\sqrt{-5}]$ the only units are ± 1. why

If x is a unit $x = a + b\sqrt{-5}$ then

$x\bar{x} = (a+b\sqrt{-5})(a-b\sqrt{-5}) = a^2 + 5b^2 = 1$

But the only integer solutions are $(a,b) = (\pm 1, 0)$.

$2, 3, 1+\sqrt{-5}, 1-\sqrt{-5} \in \mathbb{Z}[\sqrt{-5}]$ are irreducible.

why? If $2 = \alpha\beta$, $\alpha, \beta \in \mathbb{Z}[\sqrt{-5}]$ then

then $4 = |2|^2 = |\alpha|^2 |\beta|^2$.

$\alpha = a + c\sqrt{-5}$, $\beta = b + d\sqrt{-5}$

$|\alpha|^2 = (a+c\sqrt{-5})(a-c\sqrt{-5})$ $\quad |\beta|^2 = b^2 + 5d^2$

$= a^2 + 5c^2$

$4 = |\alpha|^2 |\beta|^2$ where $|\alpha|^2, |\beta|^2 \in \{1, 2, 3, 4, 5, \ldots\}$

$1 \times 4 \Rightarrow \alpha$ unit

~~2×2~~ $\quad a^2 + 5c^2 = 2$ has no solution.

$4 \times 1 \Rightarrow \beta$ is a unit.

Why does $\mathbb{Z}$ have unique factorization?

It's rather easy to show every integer factors into primes.

$12 = 3 \times 4 = 3 \times 2 \times 2$

There is no infinite decreasing sequence $n_1 > n_2 > n_3 > n_4 > \cdots > 0$ in the positive integers.

The positive integers are well-ordered. (Equivalently, use induction).

Why is the prime factorization of a positive integer n unique?

Can $n = p_1 p_2 p_3 \cdots p_r = q_1 q_2 q_3 \cdots q_s$ where $p_1, p_2, \cdots, p_r, q_1, q_2, \cdots, q_s$ are primes?

We would need to show that every prime on the left (every p_i) occurs also among $q_1, \cdots, q_s$.

Basic step: If p is a prime and $a, b \in \mathbb{Z}$ with $p \mid ab$, then show $p \mid a$ or $p \mid b$.

Euclid's Lemma If a prime p divides ab in $\mathbb{Z}$, then $p \mid a$ or $p \mid b$.

("or" is inclusive. Eg. $3 \mid 6 \cdot 9 \Rightarrow 3 \mid 6$ or $3 \mid 9$.)

$6 \mid 4 \cdot 9$ but $6 \nmid 4$, $6 \nmid 9$. We really need p to be prime.

$p \mid ab$ says: $pk = ab$ for some k. Can we argue: p is a prime factor on the left so it's a factor on the right so p is a factor in a or in b. No! This is a common fallacy.

Proof of Euclid's lemma: Suppose $p \mid ab$ ie. $pk = ab$ for some k.
Either $p \mid a$ or $p \nmid a$. If $p \mid a$ we're done. Hence we may assume $p \nmid a$.

Then $\gcd(p, a) = 1 = rp + sa$ for some $r, s \in \mathbb{Z}$. ← p not in this list

Divisors of a: $\pm 1, \cdots, \pm a$

Divisors of p: $\pm 1, \pm p$.

$b = rbp + sab$ is divisible by p since both terms rbp and sab are divisible by p. □

If $p \mid abc$ where p is prime then $p \mid a$ or $p \mid b$ or $p \mid c$.

$p \mid abc \Rightarrow p \mid a$ or $p \mid bc$. $\Rightarrow p \mid a$ or $p \mid b$ or $p \mid c$.

This argument extends to any number of factors ie. if $p \mid q_1 q_2 \cdots q_l$ where p is prime then $p \mid q_j$ for some $j \in \{1, 2, \cdots, l\}$. (To formalize this argument, use induction.)

Fundamental Theorem of Arithmetic Every positive integer has a unique factorization as a product of primes.

If $n = p_1 p_2 \cdots p_r = q_1 q_2 \cdots q_s$ where $p_1, p_2, \cdots, p_r, q_1, q_2, \cdots, q_s$ are primes then $p_1 \mid n$ so $p_1 \mid q_j$ for some $j \in \{1, 2, \cdots, s\}$ so $p_1 = q_j$. Cancel this prime factor from both sides and repeat the argument with the remaining prime factors.

On to Chapter 9.

Look at powers in $\mathbb{Z}/7\mathbb{Z} = \mathbb{F}_7 = \{0,1,2,3,4,5,6\} = \{0,\pm1,\pm2,\pm3\}$ eg. $3^2 = 9 = 2$, $\frac{1}{4} = 2$.

$-3 = 6+6+6 = 4$

x	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8
0	1	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1
2	1	2	4	1	2	4	1	2	4
3	1	3	2	6	4	5	1	3	2
$-3=4$	1	4	2	1	4	2	1	4	2
$-2=5$	1	5	4	6	2	3	1	5	4
$-1=6$	1	6	1	6	1	6	1	6	1

$x^5 = x^{-1} = \frac{1}{x}$ if $x \neq 0$

palindromic sequence $1,4,2,2,4,1$

Theorem Let p be prime. Then for all $a \in \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} = \{0,1,2,\dots,p-1\}$, $a^p = a$;
furthermore $a^{p-1} = 1$ if $a \neq 0$.

Remark We really do need p to be prime.

"Fermat's Little Theorem" can be rephrased as:
Let p be prime. Then for all $a \in \mathbb{Z}$, $a^p \equiv a \bmod p$. Moreover if $p \nmid a$ (p doesn't divide a) then $a^{p-1} \equiv 1 \bmod p$.

Solve the linear system $\begin{cases} 3x + 5y = 2 \\ 4x + 7y = 5 \end{cases}$ for $x, y \in \mathbb{F}_{11} = \{0, 1, 2, \cdots, 10\}$

$\frac{1}{3} = 4$

$$\begin{matrix} x & y \\ \end{matrix}$$
$$\left[\begin{array}{cc|c} 3 & 5 & 2 \\ 4 & 7 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 9 & 8 \\ 4 & 7 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 9 & 8 \\ 0 & 4 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 9 & 8 \\ 0 & 1 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 7 \end{array}\right]$$
$$\begin{matrix} & & x & y \\ \end{matrix}$$

$7 - 36 = 7 - 3 = 4$

$5 - 32 = 5 + 1 = 6$

$8 - 63 + 66 = 8 + 3 = 0$

The unique solution is $(x, y) = (0, 7)$.

Check: $(0, 7)$ satisfies both equations.

Computing a^p and a^{p-1} may be infeasible but computing $a^p \bmod p$ and $a^{p-1} \bmod p$ is feasible even when a, p are hundreds of digits long.

Mathematica commands `Mod[a^k, n]` computes $a^k \bmod n$. This will fail if a, k are hundreds of digits long. Instead use the command `PowerMod[a, k, n]`

"Modular exponentiation" is a very important operation in cryptography, primality testing, pseudorandom number generation, etc.

Factorization of integers is much harder than primality testing.

(believed to be) non-poly. time. — infeasible

polynomial-time — feasible

Fermat's Little Theorem Let $\mathbb{F}_p = \{0,1,2,\cdots,p-1\} = \mathbb{Z}/p\mathbb{Z}$ where p is prime. Then $a^p = a$ for all $a \in \mathbb{F}_p$; and $a^{p-1} = 1$ for all nonzero $a \in \mathbb{F}_p$.

Proof Consider the product of all nonzero elements of $\mathbb{F}_p$: $b = \prod_{0 \neq a \in \mathbb{F}_p} a = 1 \times 2 \times 3 \times \cdots \times (p-1) \in \mathbb{F}_p^\times = \{1,2,\cdots,p-1\}$
$= \{$units in $\mathbb{F}_p\}$
$= \{$nonzero elements in $\mathbb{F}_p\}$ since p is prime

(Actually $b = (p-1)!$ but this is now mod p.) Let $a \in \mathbb{F}_p^\times$ (nonzero element of $\mathbb{F}_p$).
The map $x \mapsto ax$, $\mathbb{F}_p \to \mathbb{F}_p$ is bijective. (Its inverse is $x \mapsto a^{-1}x$).

$b = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (p-1)$
$a^{p-1}b = a \cdot 2a \cdot 3a \cdot \cdots \cdot (p-1)a = b \Rightarrow a^{p-1}bb^{-1} = bb^{-1}, \Rightarrow a^{p-1} = 1.$

eg. $p=7$, $a=3$, $b = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6$
$3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18$
$3 \cdot 6 \cdot 2 \cdot 5 \cdot 1 \cdot 4 = b$

For the first conclusion, let $a \in \mathbb{F}_p$ (zero or nonzero).
If $a \neq 0$ then $a \cdot a^{p-1} = 1 \cdot a$ so $a^p = a$. If $a = 0$ then $a^p = 0 = a$. □

Fermat's Little Theorem is really a special case of Lagrange's theorem in group theory.
If H is a subgroup of a finite group G then $|H|$ divides $|G|$. In particular the order of every element of G divides $|G|$. So $g^n = 1$ for all $g \in G$ where $n = |G|$.
$\mathbb{F}_p^\times$ is a group of order $p-1$ so $a^{p-1} = 1$ for all $a \in \mathbb{F}_p^\times$.

Fermat's Little Theorem plays a huge role in primality testing.

Generalization of Fermat's Little Theorem (Euler's Formula/Theorem) allows us to work with modular exponentiation mod m, m not necessarily prime.

This requires Euler's "totient" function $\phi(m)$ = number of integers $k \in \{1, 2, \cdots, m\}$ that are relatively prime to m

"Freud"

eg. $\phi(p) = p-1$ whenever p is prime

m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\phi(m)$	1	1	2	2	4	2	6	4	6	4	10	4	12	6	8	8	16

1, 2, 3, 4, 5, 6
1, 2, 3, 4, 5
1, 2, 3, 4

$\phi(p^k) = p^k - p^{k-1} = p^k(1 - \frac{1}{p})$ (p prime, $k \geq 1$)

$\phi(mn) = \phi(m)\phi(n)$ whenever $\gcd(m, n) = 1$.

$\phi(5) = 4$
$\phi(25) = 20 \neq \phi(5)\phi(5)$
$\phi(9) \neq \phi(2)\phi(2)$

$\mathbb{Z}/15\mathbb{Z} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$
$\cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$

$\phi(15) = 8 = \phi(3)\phi(5)$

$\phi(30) = \phi(5)\phi(6) = 4 \cdot 2 = 8$

$\mathbb{Z}/30\mathbb{Z} = \{0, 1, 2, \cdots, 29\}$

$\{k \in \mathbb{Z}/30\mathbb{Z} : \gcd(k, 30) = 1\}$
$= \{1, 7, 11, 13, 17, 19, 23, 29\}$

mod 5

mod 3	0	1	2	3	4
0	0	6	12	3	9
1	10	1	7	13	4
2	5	11	2	8	14

mod 6

mod 5	0	1	2	3	4	5
0	0	25	20	15	10	5
1	6	1	26	21	16	11
2	12	7	2	27	22	17
3	18	13	8	3	28	23
4	24	19	14	9	4	29

$\mathbb{Z}/4\mathbb{Z} = \{0, 1, 2, 3\}$

mod 2

mod 2	0	1
0	0	
1		1

If R, S are rings then $R \oplus S$ (also denoted $R \times S$) is $R \oplus S = \{(r, s) : r \in R, s \in S\}$ is a ring with componentwise operations

$(r, s) + (r', s') = (r+r', s+s')$
$(r, s)(r', s') = (rr', ss')$

$(r, s) \in R \oplus S$ is a unit iff r is a unit in R and s " " " S.

For example we look for integer solutions of some Diophantine equation e.g. $61x^2 - y^2 = 31$. Does this have integer solutions? If so, show by example. If not (i.e. no solutions in $\mathbb{Z}$) then find $m \in \mathbb{Z}$ such that the equation has no solutions in $\mathbb{Z}/m\mathbb{Z}$. In this case it's best to consider prime powers $m = p^k$, p prime, $k \geq 1$.

$x^2 - 3y^2 = 3$ has solutions mod 2 but no solutions mod 4. (So no solutions in $\mathbb{Z}$)
$x^2 + y^2 + z^2 = 7$ has solutions mod 4 but no solutions mod 8. (- - - - -)

Euler's Formula If a, m are integers, $m \geq 1$ and $\gcd(a,m) = 1$, then $a^{\phi(m)} \equiv 1 \bmod m$.

This generalizes Fermat's Little Theorem: if p is prime and $\gcd(a,p) = 1$ then $a^{\phi(p)} \equiv 1 \bmod p$ so $a^{p-1} \equiv 1 \bmod p$.

The proof of Euler's Formula is the same as the proof of Fermat's Little Theorem.
Both are special cases of Lagrange's Theorem for finite groups.

Eg. Find the last two decimal digits of 1234567^{531}

i.e. find $r \in \{0, 1, 2, \ldots, 99\}$ such that $1234567^{531} \equiv r \bmod 100$.

$1234567^{531} \equiv 67^{531} \bmod 100$

67 is relatively prime to 100 so $67^{\phi(100)} \equiv 1 \bmod 100$ where $\phi(100) = \phi(2^2 \cdot 5^2) = \phi(2^2)\phi(5^2)$

$= (2^2 - 2)(5^2 - 5) = 2 \times 20 = 40$ so $67^{40} \equiv 1 \bmod 100$ $\qquad 531 = 13 \times 40 + 21$

$$67^{531} = 67^{13 \times 40 + 21} = \underbrace{(67^{40})^{13}}_{1} \cdot 67^{21} \equiv 67^{21} \bmod 100$$

mod 5

mod 3 \	0	1	2	3	4
0	0	6	12	3	9
1	10	1	7	13	4
2	5	11	2	8	14

Solving a pair of congruences

If $\gcd(m,n)=1$ then the system $\begin{cases} x \equiv a \bmod m \\ x \equiv b \bmod n \end{cases}$

has a unique solution mod mn.

eg. solve $\begin{cases} x \equiv 1 \bmod 3 \\ x \equiv 3 \bmod 5 \end{cases}$ $\iff$ $x \equiv 13 \bmod 15$.

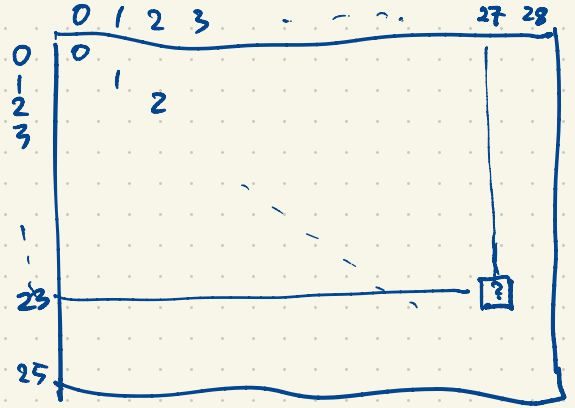
$\Leftarrow$ is trivial. Given $x \equiv 13 \bmod 15$, we compute

$x \equiv 13 \equiv 1 \bmod 3$

$x \equiv 13 \equiv 3 \bmod 5$.

$\Rightarrow$ takes a little more work.

eg. $m = 26$, $n = 29$ Solve $\begin{cases} x \equiv 23 \bmod 26 \\ x \equiv 27 \bmod 29 \end{cases}$ $\iff$ $x \equiv ? \bmod \underbrace{754}_{26 \cdot 29}$



Answer without completing all 754 entries in the table.

$x \equiv 23 \bmod 26$

$x = 26k + 23 \equiv 27 \bmod 29$

$26k \equiv 4 \bmod 29$

Find the inverse of 26 mod 29.

$67^{21} \equiv ?$ mod 100

$67^1 \equiv 67$

square → $67^2 \equiv 89$ mod 100 $\qquad 67^2 = 4489 \equiv 89$ mod 100

square → $67^4 \equiv 89^2 \equiv 21$ ...

square → $67^8 \equiv 21^2 \equiv 41$ ...

square → $67^{16} \equiv 41^2 \equiv 81$ ....

$67^{21} = 67^{16} \cdot 67^4 \cdot 67^1 \equiv 81 \cdot 21 \cdot 67 \equiv 67$ mod 100

So $1234567^{531} \equiv 67$ mod 100.

decimal 2025 written in binary?

2 | 2025		
2 | 1012	r 1	
2 | 506	r 0	
2 | 253	r 0	
2 | 126	r 1	
2 | 63	r 0	
2 | 31	r 1	
2 | 15	r 1	
2 | 7	r 1	
2 | 3	r 1	
2 | 1	r 1	
0	r 1	

$2025_{10} = 11111101001_{two}$

Fast exponentiation using binary representation of the exponent: $21 = 16 + 4 + 1$

i.e. the decimal number 21 is 10101

16's, 8's, 4's, 2's, 1's

i.e. $2025 = 1024 + 512 + 256 + 128 + 64 + 32 + 8 + 1$

$x^{2025} = x^{1024} \cdot x^{512} \cdot x^{256} \cdot x^{128} \cdot x^{64} \cdot x^{32} \cdot x^8 \cdot x^1$

Computing a^k mod m can be done in polynomial time.

See Chapter 16: ... repeated squaring.