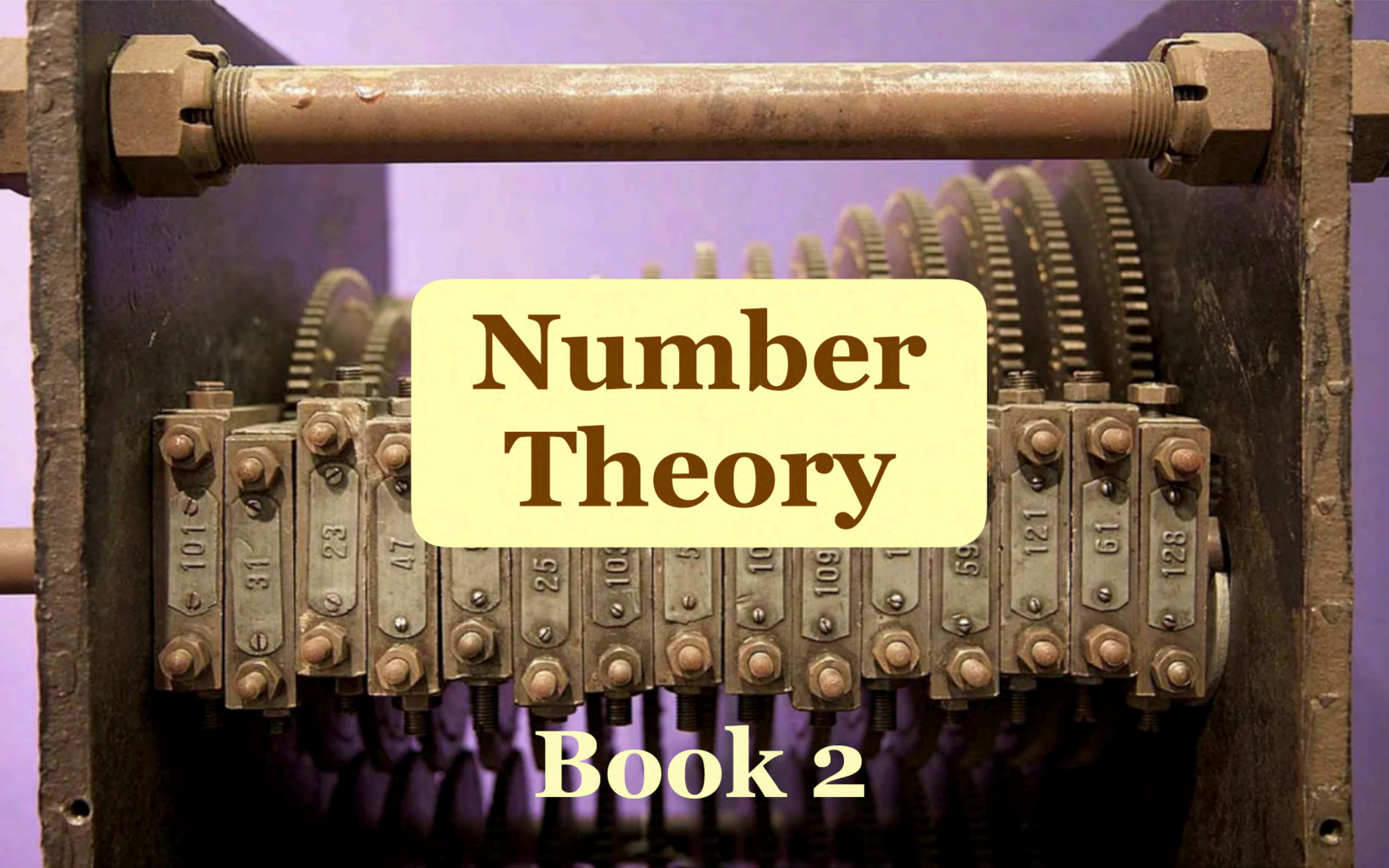

# Number Theory

## Book 2

"Trick" (or technique) for identifying which small integers have the form $x^2+y^2$ or $x^2+3y^2$ or ...

Define $\theta(t) = \sum_{n=-\infty}^{\infty} t^{n^2} = 1 + 2t + 2t^4 + 2t^9 + 2t^{16} + 2t^{25} + \cdots = \sum_{n\in\mathbb{Z}} t^{n^2}$

$\theta(t)^2 = (1+2t+2t^4+2t^9+2t^{16}+2t^{25}+\cdots)(1+2t+2t^4+2t^9+2t^{16}+2t^{25}+\cdots) = 1+4t+4t^2+4t^4+8t^5+4t^8+\cdots+12t^{25}+\cdots$

The powers of $t$ appearing in this expansion are precisely the exponents expressible as a sum of two squares.

The coefficient of $t^n$ on the right hand side is the number of solutions of $n = x^2+y^2$ $(x,y\in\mathbb{Z})$

eg. $25 = x^2+y^2$ has 12 solutions $(\pm5,0)$, $(0,\pm5)$, $(\pm3,\pm4)$, $(\pm4,\pm3)$

$5 = x^2+y^2$ has 8 solutions $(\pm2,\pm1)$, $(\pm1,\pm2)$

$6 = x^2+y^2$ has 0 solutions.

$\theta(t)^3 = 1+6t+12t^2+8t^3+6t^4+24t^5+24t^6+12t^8+\cdots$

$1 = x^2+y^2+z^2$ has six solutions $(\pm1,0,0)$, $(0,\pm1,0)$, $(0,0,\pm1)$

$2 =$ ... twelve $(\pm1,\pm1,0)$, $(\pm1,0,\pm1)$, $(0,\pm1,\pm1)$

$7 = x^2+y^2+z^2$ has 0 solutions

$\theta(t)^4$ has positive coefficient of $t^n$ for every positive integer $n$

Lagrange's Theorem: every positive integer is a sum of four squares

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Fundamental Theorem of Arithmetic: Every positive integer is uniquely expressible as a product of primes.

We'll explain exactly what this says and we'll prove it.

Fundamental Theorem of Calculus

" " of Linear Algebra

" " " Algebra