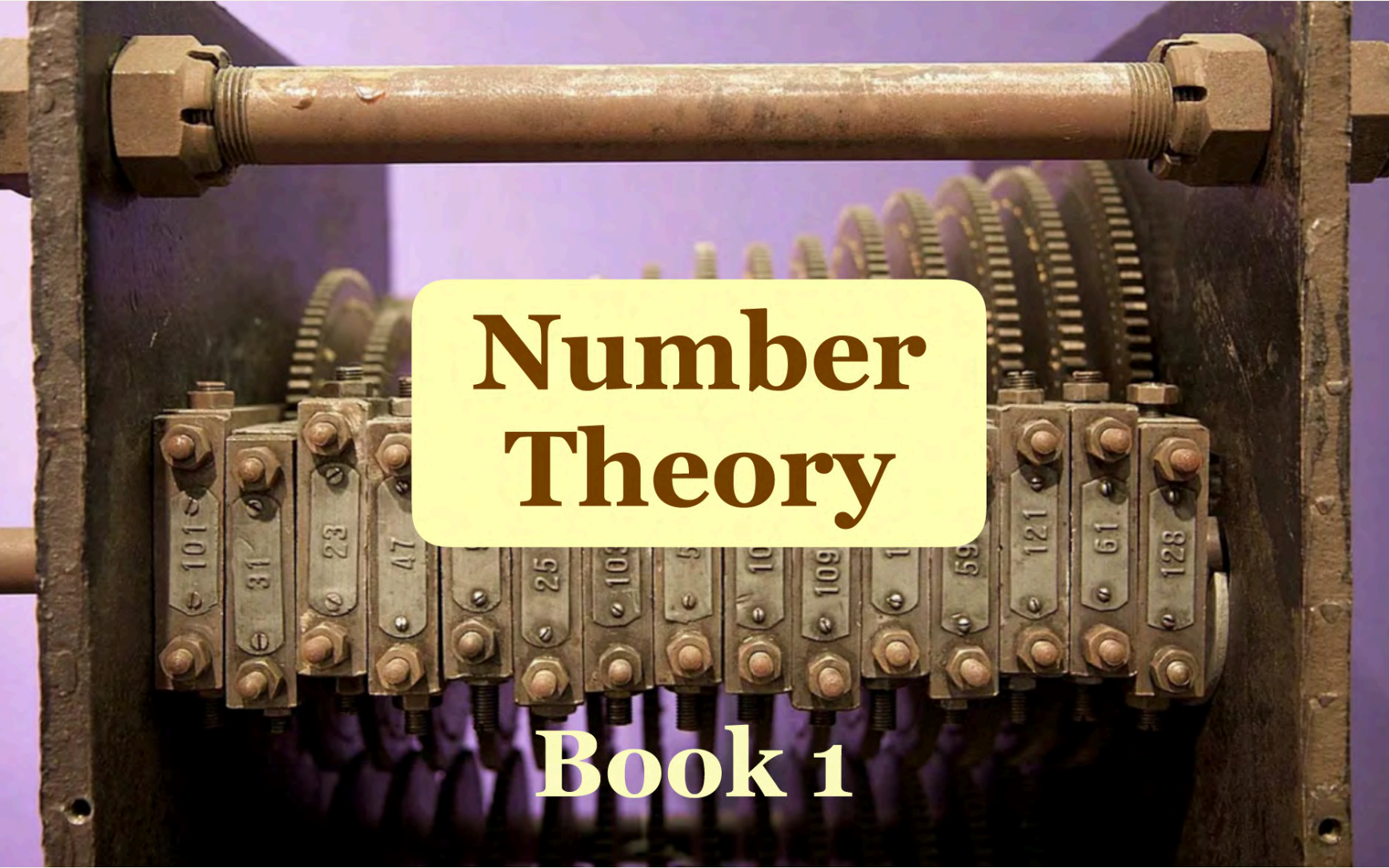
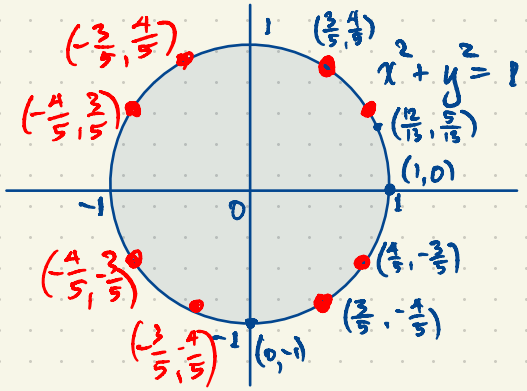

Number
Theory
Book 1



How many points on the circle $x^2+y^2=1$ $(x,y\in\mathbb{Q})$ have rational number coordinates?

Not $(\frac{1}{2},\pm\frac{\sqrt{3}}{2})$

Are there infinitely many "rational points" on the unit circle?

$(\frac{3}{5},\frac{4}{5}) \longleftrightarrow 3^2+4^2=5^2$ solution of $x^2+y^2=z^2$ $(x,y,z\in\mathbb{Z})$

A Pythagorean triple is a triple (a,b,c) of positive integers a,b,c, satisfying $a^2+b^2=c^2$.

eg. $(3,4,5)$, $(6,8,10)$, $(9,12,15)$, $(5,12,13)$, ...

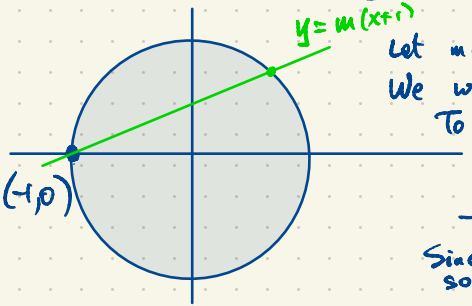
A triple (a,b,c) is primitive if it is not an integer scalar multiple of a smaller triple eg. $(3,4,5)$ is primitive; $(6,8,10)=2(3,4,5)$ is imprimitive, as is $(9,12,15)=3(3,4,5)$.

There are infinitely many primitive Pythagorean triples.

The triple $(3,4,5)$ yields eight rational points $(\pm\frac{3}{5},\pm\frac{4}{5})$, $(\pm\frac{4}{5},\pm\frac{3}{5})$. So does $(4,3,5)$

Theorem There are infinitely many rational points on the unit circle $x^2+y^2=1$. See Chapter 3.

Proof



Let $m\in\mathbb{Q}$. Consider the line $y=m(x+1)$ through $(-1,0)$.

We will see that this line intersects the circle in two rational points.

To find these points, solve $\begin{cases} y=m(x+1) \\ x^2+y^2=1 \end{cases}$ for (x,y).

$x^2+\big(m(x+1)\big)^2=1$ (we have eliminated y from this equation)

This is a quadratic equation in x with rational coefficients.

Since $x=-1$ is one rational root, the other root must also be rational so (x,y) is rational. Every $m\in\mathbb{Q}$ gives a rational point on the unit circle.

Starting over, we give a completely algebraic approach to parameterizing the primitive Pythagorean triples.

$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ ring of integers.

$\mathbb{N} = \{1, 2, 3, 4, 5, \dots\}$ positive integers.

$\mathbb{N}$ has unique factorization. Every $n \in \mathbb{N}$ factors uniquely as a product of prime numbers $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, \dots$

i.e. if $n = p_1 p_2 \cdots p_k = q_1 q_2 \cdots q_\ell$ where all p_i, q_j are primes then $k = \ell$ and $p_i = q_i$ after re-indexing if necessary.

eg. $12 = 2 \times 6 = 2 \times 2 \times 3$ is a prime factorization of 12.

$12 = 3 \times 4 = 3 \times 2 \times 2$ " " " "

$1 = 1$ is a prime factorization with 0 prime factors.

A prime number is an integer $n > 1$ which is not of the form ab $(a, b \in \mathbb{N},\ a, b > 1)$.

We'll assume unique factorization for now but later, we'll have to explain this.

$\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$

x	x^2
0	0
1	1

$\gcd(a, b)$ = greatest common divisor of a, b

for $a, b \in \mathbb{N}$ eg. $\gcd(40, 68) = 2 \times 2 = 4$.

($40 = 2 \times 2 \times 2 \times 5$, $68 = 2 \times 2 \times 17$)

$(3, 4, 5)$, $(4, 3, 5)$

Pythagorean triple (a, b, c), a, b, c positive integers with $a^2 + b^2 = c^2$

(a, b, c) is primitive if it's not a scalar multiple (ka', kb', kc') with $k > 1$. $(6, 8, 10) = 2(3, 4, 5)$ is imprimitive.

If (a, b, c) is a primitive Pythagorean triple, what can we say about the parity of a, b, c?

parity — the quality of being even or odd

a, b, c can't all be even and they can't all be odd. In fact one must be even and the other two must be odd.

Can a, b be odd and c even? No.

Integers mod 4 $\mathbb{Z}/4\mathbb{Z} = \{0, 1, 2, 3\}$

x	x^2
0	0
1	1
2	0
3	1

If a, b are odd then $a^2 + b^2 \equiv 2 \bmod 4$ but if c is even then $c^2 \equiv 0 \bmod 4$

There is no Pythagorean triple (a, b, c) with a, b odd.

So every primitive Pythagorean triple is either (even, odd, odd) or (odd, even, odd).

Without loss of generality, take (odd, even, odd) ie. a, c odd, b even.

We will prove:

Theorem Every primitive Pythagorean triple has the form $(a,b,c) = (m^2-n^2,\ 2mn,\ m^2+n^2)$ for a unique pair of relatively prime integers m, n of opposite parity (i.e. one even, the other odd) with $m > n \geq 1$. (Or with a,b reversed.)
Every such triple is a primitive Pythagorean triple.

Towards the proof, let's observe that in a primitive Pythagorean triple (a,b,c), any two of a,b,c are relatively prime i.e. $\gcd(a,b) = 1 = \gcd(a,c) = \gcd(b,c)$. Why?

Suppose (a,b,c) is not primitive, i.e. $(a,b,c) = (ka', kb', kc')$ with $k \geq 2$. Then $\gcd(a,b) \neq 1$ $(\gcd(a,b) \geq k)$
$\gcd(a,c) \neq 1$
$\gcd(b,c) \neq 1$.

Suppose (a,b,c) is a primitive Pythagorean triple. Why must $\gcd(a,b) = 1$?
Why must $\gcd(a,c) = 1$?
Why must $\gcd(b,c) = 1$?