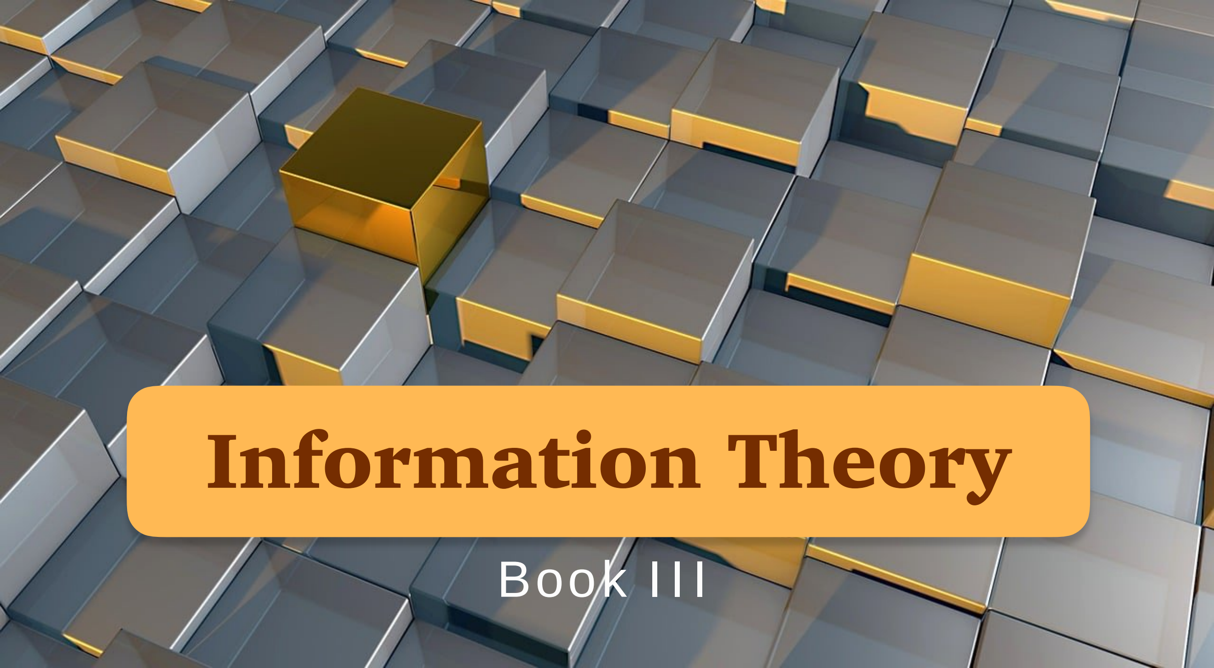


Information Theory

Book III

Spin state of an electron (disregard position and momentum) is an example of a qubit, which is a vector $|\psi\rangle \in \mathbb{C}^2 = \{ \binom{\alpha}{\beta} : \alpha, \beta \in \mathbb{C} \}$.

ket notation

$|\psi\rangle = \binom{\alpha}{\beta} = \alpha|+\rangle + \beta|-\rangle$, $\quad |\alpha|^2 + |\beta|^2 = 1$.

Standard basis of $\mathbb{C}^2$: $|+\rangle = \binom{1}{0}$, $|-\rangle = \binom{0}{1}$
"spin up" "spin down"

An electron in this spin state is in a superposition of spin up and spin down states.

A linear functional on $\mathbb{C}^2$ is a linear transformation
$\langle\phi| : \mathbb{C}^2 \to \mathbb{C}$
bra notation

$\langle\phi| = (\gamma\ \delta) : \binom{\alpha}{\beta} \mapsto (\gamma\ \delta)\binom{\alpha}{\beta} = \underbrace{\gamma\alpha + \delta\beta}_{= \langle\phi|\psi\rangle} \in \mathbb{C}$.

Dual basis:

$\langle +| = |+\rangle^* = (1\ 0)$ (conjugate transpose)

$\langle -| = |-\rangle^* = (0\ 1)$

$|\psi\rangle^* = \binom{\alpha}{\beta}^* = (\bar{\alpha}\ \bar{\beta}) = \bar{\alpha}\langle +| + \bar{\beta}\langle -|$

$\langle +|\psi\rangle = \langle +|(\alpha|+\rangle + \beta|-\rangle) = \alpha$

$\langle -|\psi\rangle = \beta$.

A measurement of an electron in this spin state yields a single bit of classical information:

- spin up, with probability $|\alpha|^2$;
- spin down, with probability $|\beta|^2$.

This says what happens when we measure with respect to the z-axis. (For measurement in a different direction/axis, we'll say later.)

As soon as the measurement is taken, the spin state collapses; all knowledge of α, β is then lost.

Spin states are unit vectors in $\mathbb{C}^2$ i.e. $\binom{\alpha}{\beta}$, $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$.
i.e. in $\mathbb{R}^4$
so $|\psi\rangle \in S^3$ = unit sphere in $\mathbb{R}^4$.

$\left.\begin{array}{l}\alpha = \alpha_1 + \alpha_2 i\\ \beta = \beta_1 + \beta_2 i\end{array}\right\}$ $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$

$\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 = 1$

Any time we measure a spin state $|\psi\rangle \in S^3$, it collapses.
But it is possible to perform certain reversible operations $|\psi\rangle \mapsto A|\psi\rangle$ where A is a 2×2 unitary matrix $(AA^* = A^*A = I = \begin{pmatrix}1&0\\0&1\end{pmatrix})$ over $\mathbb{C}$.

Special examples of unitary matrices are scalar matrices $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \lambda I$, $\lambda \in \mathbb{C}$, $|\lambda| = 1$

These perform an operation on $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ whose only effect is to alter the phase of α, β

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \longmapsto A|\psi\rangle = \begin{pmatrix} \lambda\alpha \\ \lambda\beta \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$\lambda = e^{i\theta}$ $\quad (\theta \in [0, 2\pi))$

which has no physical significance. For this reason the so-called density matrix

$$\underbrace{\underbrace{|\psi\rangle}_{2\times 1} \underbrace{\langle\psi|}_{1\times 2}}_{2\times 2} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\beta} \end{pmatrix} = \begin{pmatrix} \alpha\bar{\alpha} & \alpha\bar{\beta} \\ \beta\bar{\alpha} & \beta\bar{\beta} \end{pmatrix}$$

which holds all the physically significant information of the single qubit.

Hermitian 2×2 matrix

$H \in \mathbb{C}^{2\times 2}$ $\quad$ (2×2 complex matrix)

satisfying $H^* = H$

The map $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \longmapsto \begin{pmatrix} \lambda\alpha \\ \lambda\beta \end{pmatrix}$ does not change this density matrix.

Entanglement typically occurs when we include multiple electrons in our system.

Start by reviewing statistical dependence works:

Let's say we take a random individual A from a population.

Imagine the population is 40% male, 60% female; 30% short, 70% tall.

Sampling by selecting one person gives two bits: MS, MT, FS, or FT.

Combinations of attributes: 12%, 28%, 18%, 42% if gender is independent of height.

Height

Gender	S	T	
M	0.12	0.28	0.4
F	0.18	0.42	0.6
	0.3	0.7	1

In this example, gender and height are independent.

$$\begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} \begin{bmatrix} 0.3 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.12 & 0.28 \\ 0.18 & 0.42 \end{bmatrix}$$

Outer product of two vectors. is a rank 1.

More typical distribution

	S	T	
M	0.1	0.3	0.4
F	0.2	0.4	0.6
	0.3	0.7	1

The matrix $\begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.4 \end{bmatrix}$ has rank 2.

In this second example gender and height are (statistically) dependent.

If one electron has (spin) state $|\psi_1\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \in \mathbb{C}^2$ and a second electron has spin state $|\psi_2\rangle = \begin{pmatrix}\gamma\\ \delta\end{pmatrix} \in \mathbb{C}^2$

$|\alpha|^2+|\beta|^2=1$ $\qquad$ $|\gamma|^2+|\delta|^2=1$

the pair of electrons has state $|\psi_{12}\rangle = \alpha_{11}|{+}{+}\rangle + \alpha_{12}|{+}{-}\rangle + \alpha_{21}|{-}{+}\rangle + \alpha_{22}|{-}{-}\rangle \in \mathbb{C}^4$

$\alpha_{ij} \in \mathbb{C}$, $\quad |\alpha_{11}|^2 + |\alpha_{12}|^2 + |\alpha_{21}|^2 + |\alpha_{22}|^2 = 1.$

prob. of both electrons having spin up

If the two electrons are <u>not</u> entangled then

$$\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix} = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}\begin{pmatrix}\gamma & \delta\end{pmatrix} \qquad \text{rank } 1.$$

If the matrix has rank 2 then the two electrons are entangled.

$$\left.\begin{aligned} &\text{Eg. } |\psi\rangle = \tfrac{1}{\sqrt2}\left(|{+}{+}\rangle + |{-}{-}\rangle\right) \quad \text{i.e. } \begin{pmatrix}\frac{1}{\sqrt2} & 0\\ 0 & \frac{1}{\sqrt2}\end{pmatrix}\\ &|\psi'\rangle = \tfrac{1}{\sqrt2}\left(|{+}{-}\rangle + |{-}{+}\rangle\right) \quad \text{i.e. } \begin{pmatrix}0 & \frac{1}{\sqrt2}\\ \frac{1}{\sqrt2} & 0\end{pmatrix}\end{aligned}\right\} \text{Examples of EPR pairs}$$

One way to talk about the spin state of a set of n electrons is

$$|\psi\rangle = \sum_{\substack{i_1\in\{0,1\}\\ i_2\in\{0,1\}\\ \vdots\\ i_n\in\{0,1\}}} \alpha_{i_1 i_2 i_3 \cdots i_n} |\underbrace{\pm\pm\pm\cdots\pm}_{\text{all } 2^n \text{ combinations of } \pm}\rangle \in \mathbb{C}^{2^n} \qquad \sum |\alpha_{i_1 i_2\cdots i_n}|^2 = 1$$

$\left(\alpha_{i_1 i_2 \cdots i_n} : i_1, i_2, \ldots, i_n \in \{0,1\}\right)$ is a $\underbrace{2\times2\times2\times\cdots\times2}_{n}$ array or tensor

$\mathbb{C}^{2^n} = \underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n \text{ times}}$ tensor product. Take basis $|+\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|-\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

has basis
$$|{+}{+}{+}\cdots{+}{+}\rangle = |+\rangle \otimes |+\rangle \otimes \cdots \otimes |+\rangle$$
$$|{+}{+}{+}\cdots{+}{-}\rangle = |+\rangle \otimes |+\rangle \otimes \cdots \otimes |-\rangle$$
$$\vdots$$
$$|{-}{-}{-}\cdots{-}\rangle = |-\rangle \otimes |-\rangle \otimes \cdots \otimes |-\rangle .$$

More generally if $v_i \in \mathbb{C}^2$ $(i = 1, 2, \dots, n)$
then $v_1 \otimes v_2 \otimes \cdots \otimes v_n \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$. (pure tensors simple)

$$\mathbb{C}^2 \times \mathbb{C}^2 \times \cdots \times \mathbb{C}^2 \qquad \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 = \mathbb{C}^{2^n}$$
$$\rotatebox{90}{$\in$} \qquad\qquad \rotatebox{90}{$\in$}$$
$$(v_1, \cdots, v_n) \longmapsto v_1 \otimes v_2 \otimes \cdots \otimes v_n$$
this map is multilinear
i.e. linear in each argument separately.

In $\mathbb{C}^m \otimes \mathbb{C}^n \cong \mathbb{C}^{mn}$
every vector is a sum of at most $\min\{m, n\}$ pure tensors.
The corresponding result for $\mathbb{C}^{m_1} \otimes \mathbb{C}^{m_2} \otimes \cdots \otimes \mathbb{C}^{m_k}$ is not known and extremely hard.
(In Algebraic Geometry look up Higher Secant varieties of Segre Varieties)

Bell's Theorem
Gleason's Theorem
Kochen-Specker Theorem

Recall: the spin state of a single electron is a cubit $|\psi\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \in \mathbb{C}^2$, $|\alpha|^2 + |\beta|^2 = 1$.

Standard basis $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

spin up/down with respect to the z-axis

How do we measure the spin in an arbitrary direction?

In the vertical direction we make use of basis $|+_z\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|-_z\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ basis of eigenvectors for the Pauli spin operator $\sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$.

$$\sigma_z |+_z\rangle = |+_z\rangle$$

$$\sigma_z |-_z\rangle = -|-_z\rangle$$

Any electron with spin state $|\psi\rangle = \alpha|+_z\rangle + \beta|-_z\rangle$ can be measured in the vertical direction

$$\sigma_z |\psi\rangle = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\alpha\\ -\beta\end{pmatrix}$$

Follow this by a linear functional eg. $|+_z\rangle^* = \langle +_z|$

$$\langle +_z|\sigma_z|\psi\rangle = (1\ 0)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = (1\ 0)\begin{pmatrix}\alpha\\ -\beta\end{pmatrix} = \alpha,$$ the amplitude for the electron to be spin up.

Once the measurement is performed, the state collapses into that spin state $|\psi\rangle \longmapsto |+_z\rangle$.

$$\langle +_z|\underbrace{\sigma_z|+_z\rangle}_{|+_z\rangle} = 1.$$

If we measure $|\psi\rangle \longmapsto \langle +_z|\sigma_z|\psi\rangle$ and find spin down, the state collapses to spin down $|-_z\rangle$

$$\langle +_z|\sigma_z|\psi\rangle = -\beta, \quad |-\beta|^2 = |\beta|^2$$

$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ Eigenvectors: $|+_x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $|-_x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ $|+_y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$, $|-_y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$

$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $|+_z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|-_z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ eigenvalues $+1, -1$

Hermitian: $\sigma^* = \sigma$

eg. $\sigma_y \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ i \end{pmatrix} = |+_y\rangle$

$\sigma_y \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ i \end{pmatrix} = -\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = -|-_y\rangle$

If we measure an electron having spin $|+_y\rangle$ (in the pos. y-direction) with respect to the x-axis

$\langle +_x | +_y \rangle = \frac{1}{\sqrt{2}}(1 \quad 1)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+i \\ 1+i \end{pmatrix}$ $\left|\frac{1+i}{2}\right|^2 = \frac{2}{4} = \frac{1}{2}$ $|a+bi|^2 = a^2+b^2$

Density matrix of $|\psi\rangle$ is $|\psi\rangle\langle\psi| = |\psi\rangle|\psi\rangle^* = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}(\bar{\alpha} \quad \bar{\beta}) = \begin{pmatrix} \alpha\bar{\alpha} & \alpha\bar{\beta} \\ \bar{\alpha}\beta & \beta\bar{\beta} \end{pmatrix}$ $|\alpha|^2+|\beta|^2=1$

is Hermitian having eigenvalues 1, 0 ; corresponding eigenvectors $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, $|\psi'\rangle = \begin{pmatrix} \bar{\beta} \\ -\bar{\alpha} \end{pmatrix}$

$|\psi\rangle\langle\psi|\psi\rangle = |\psi\rangle$ since $\langle\psi|\psi\rangle = 1$ $\langle\psi'|\psi\rangle = (\beta \ -\alpha)\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha\beta - \alpha\beta = 0.$

$|\psi\rangle\langle\psi|\psi'\rangle = 0. = 0|\psi'\rangle.$ $\langle\psi|\psi\rangle = 1 = \operatorname{tr} |\psi\rangle\langle\psi|$

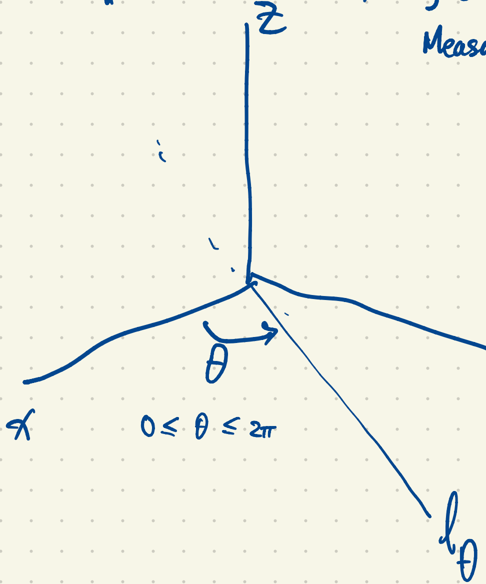
$\operatorname{tr} AB = \operatorname{tr} BA.$

What is the corresponding Pauli spin operator in an arbitrary direction $n = (n_x, n_y, n_z) \in \mathbb{R}^3$

$n_x^2 + n_y^2 + n_z^2 = 1.$

$\sigma_n = n \cdot \sigma = n_x\sigma_x + n_y\sigma_y + n_z\sigma_z$

$\sigma = (\sigma_x, \sigma_y, \sigma_z)$



Measure spin wrt line l_θ in x-y plane at angle θ as shown.

$n = (\cos\theta, \sin\theta, 0)$

σ_θ : Pauli spin operator for the direction n

$$\sigma_\theta = n \cdot (\sigma_x, \sigma_y, \sigma_z) = \cos\theta \underbrace{\sigma_x}_{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} + \sin\theta \underbrace{\sigma_y}_{\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}} = \begin{bmatrix} 0 & \cos\theta - i\sin\theta \\ \cos\theta + i\sin\theta & 0 \end{bmatrix} = \begin{bmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{bmatrix}$$

using de Moivre's formula $e^{i\theta} = \cos\theta + i\sin\theta$

Eigenvectors $|+_\theta\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} e^{-i\theta/2} \\ e^{i\theta/2} \end{bmatrix}$

$|-_\theta\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} e^{-i\theta/2} \\ -e^{i\theta/2} \end{bmatrix}$

Check: $\sigma_\theta|+_\theta\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{bmatrix}\begin{bmatrix} e^{-i\theta/2} \\ e^{i\theta/2} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} e^{-i\theta/2} \\ e^{i\theta/2} \end{bmatrix} = |+_\theta\rangle$ eigenvector with eigenvalue $+1$

$\sigma_\theta|-_\theta\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{bmatrix}\begin{bmatrix} e^{-i\theta/2} \\ -e^{i\theta/2} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -e^{-i\theta/2} \\ e^{i\theta/2} \end{bmatrix} = -|-_\theta\rangle$ " — — — — — -1.

The map $l_\theta \longmapsto \begin{cases} |+_\theta\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} e^{-i\theta/2} \\ e^{i\theta/2} \end{bmatrix} \\ |-_\theta\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} e^{-i\theta/2} \\ -e^{i\theta/2} \end{bmatrix} \end{cases}$

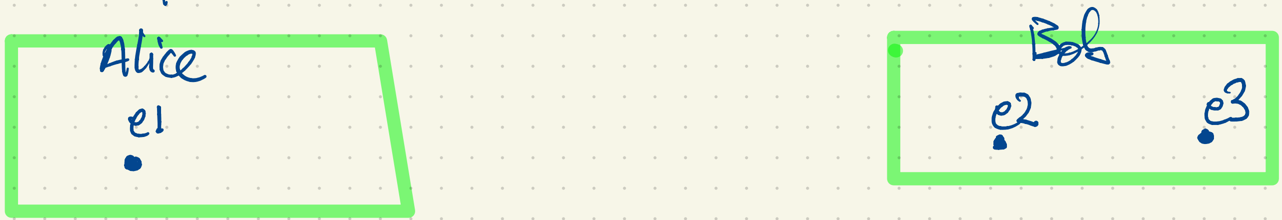
is 2-to-1.

Spin vectors go around "full circle" in $\mathbb{C}^2$ as θ goes from 0 to 4π; the "+" direction of l_θ goes twice around a circle in this same θ-interval.

If we measure an electron in spin state $|\psi\rangle = \alpha|+_\theta\rangle + \beta|-_\theta\rangle$ with respect to the direction l_θ, we get $|+_\theta\rangle$ with prob. $|\alpha|^2$, spin $|-_\theta\rangle$ with prob. $|\beta|^2$.

Spin states actually lie in S^3 = unit vector in $\mathbb{C}^2$ which is a double cover of $SO_3(\mathbb{R})$ = { rotations of $\mathbb{R}^3$ about the origin } = { 3×3 real matrices A : $AA^T = I$, $\det A = 1$ }.

Bob has an electron in spin state $|\psi\rangle = \alpha|+\rangle + \beta|-\rangle = \binom{\alpha}{\beta} \in \mathbb{C}^2$, $|\alpha|^2 + |\beta|^2 = 1$.
He wants to send this to Alice. Bob doesn't know α, β and he cannot directly measure them.
Analogy: transporting Captain Kirk from enterprise to planet's surface.
In advance of this teleportation process, Alice and Bob have stockpiled some EPR pairs



Electrons e1, e2 are entangled: their joint spin state $|\psi_{12}\rangle = \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle)$

$|++\rangle = |+\rangle \otimes |+\rangle$
$|--\rangle = |-\rangle \otimes |-\rangle$
e1 e2

Electron e3 is in state $|\psi_3\rangle = \binom{\alpha}{\beta} = \alpha|+\rangle + \beta|-\rangle$, $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$.
e3 is not (currently) entangled with e1, e2.
The combined state of e1, e2, e3 is

$$|\psi_{123}\rangle = |\psi_{12}\rangle \otimes |\psi_3\rangle = \tfrac{1}{\sqrt{2}}(|++\rangle + |--\rangle) \otimes (\alpha|+\rangle + \beta|-\rangle) \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^8$$
$$= \tfrac{1}{\sqrt{2}}(\alpha|+++\rangle + \beta|++-\rangle + \alpha|--+\rangle + \beta|---\rangle)$$

$$\left|\tfrac{\alpha}{\sqrt{2}}\right|^2 + \left|\tfrac{\beta}{\sqrt{2}}\right|^2 + \left|\tfrac{\alpha}{\sqrt{2}}\right|^2 + \left|\tfrac{\beta}{\sqrt{2}}\right|^2 = 1.$$

$|+\rangle = \binom{1}{0} \in \mathbb{C}^2$
$|-\rangle = \binom{0}{1} \in \mathbb{C}^2$

Spin of the pair e1, e2 lives in $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$ which has orthonormal basis $|++\rangle, |+-\rangle, |-+\rangle, |--\rangle$

$$|\psi_{123}\rangle = \tfrac{1}{\sqrt2}\left(\alpha|{+}{+}{+}\rangle + \beta|{+}{+}{-}\rangle + \alpha|{-}{-}{+}\rangle + \beta|{-}{-}{-}\rangle\right)$$

Bob performs a reversible (unitary) transformation with respect to e2, e3 defined by

$$|{+}{+}\rangle \mapsto \tfrac{1}{\sqrt2}(|{+}{+}\rangle + |{-}{-}\rangle)$$
$$|{-}{-}\rangle \mapsto \tfrac{1}{\sqrt2}(|{+}{+}\rangle - |{-}{-}\rangle)$$
$$|{+}{-}\rangle \mapsto \tfrac{1}{\sqrt2}(|{+}{-}\rangle + |{-}{+}\rangle)$$
$$|{-}{+}\rangle \mapsto \tfrac{1}{\sqrt2}(|{+}{-}\rangle - |{-}{+}\rangle)$$

This transforms $|\psi_{123}\rangle$ to

$$|\psi_{123}\rangle \mapsto \tfrac12\left[\left(\alpha|{+}{+}{+}\rangle + \alpha|{+}{-}{-}\rangle\right) + \left(\beta|{+}{+}{-}\rangle + \beta|{+}{-}{+}\rangle\right) + \left(\alpha|{-}{+}{-}\rangle - \alpha|{-}{-}{+}\rangle\right) + \left(\beta|{-}{+}{+}\rangle - \beta|{-}{-}{-}\rangle\right)\right]$$
$$= (\alpha|+\rangle + \beta|-\rangle)\otimes \tfrac12|{+}{+}\rangle + (\alpha|+\rangle - \beta|-\rangle)\otimes \tfrac12|{-}{-}\rangle + (\beta|+\rangle + \alpha|-\rangle)\otimes\tfrac12|{+}{-}\rangle + (\beta|+\rangle - \alpha|-\rangle)\otimes\tfrac12|{-}{+}\rangle$$

Now Bob measures e2, e3 with respect to the basis $|{+}{+}\rangle, |{+}{-}\rangle, |{-}{+}\rangle, |{-}{-}\rangle$ of $\mathbb{C}^2\otimes\mathbb{C}^2 = \mathbb{C}^4$. e2, e3 collapse into one of these four states. At this moment we know e1 is in one of the four states

$\begin{bmatrix}1&0\\0&1\end{bmatrix}: \binom{\alpha}{\beta} \mapsto \binom{\alpha}{\beta} = \alpha|+\rangle + \beta|-\rangle$

$\begin{bmatrix}1&0\\0&-1\end{bmatrix}: \binom{\alpha}{\beta} \mapsto \binom{\alpha}{-\beta}: \alpha|+\rangle - \beta|-\rangle$

$\begin{bmatrix}0&1\\1&0\end{bmatrix}: \binom{\alpha}{\beta} \mapsto \binom{\beta}{\alpha} = \beta|+\rangle + \alpha|-\rangle$

$\begin{bmatrix}0&-1\\1&0\end{bmatrix}: \binom{\alpha}{\beta} \mapsto \binom{\beta}{-\alpha} = \beta|+\rangle - \alpha|-\rangle$

Bob sends this classical information (2 classical bits) to Alice.
Alice applies the appropriate unitary 2×2 matrix to e1 which transforms e1 into the correct state.

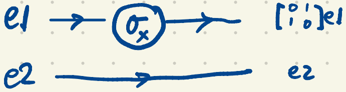
Note: Alice's operations on e1 can be described using Pauli spin matrices.

$\begin{bmatrix}1&0\\0&1\end{bmatrix}$ identity $|0\rangle\mapsto|0\rangle$, $|1\rangle\mapsto|1\rangle$ $\quad |0\rangle = \binom{1}{0}$, $|1\rangle = \binom{0}{1}$

$\sigma_x = \begin{bmatrix}0&1\\1&0\end{bmatrix}: \binom{\alpha}{\beta} \mapsto \binom{\beta}{\alpha}$ i.e. $|0\rangle \leftrightarrow |1\rangle$ 'bit flip' or 'NOT'

$\sigma_z = \begin{bmatrix}1&0\\0&-1\end{bmatrix}: \binom{\alpha}{\beta} \mapsto \binom{\alpha}{-\beta}$ i.e. $|0\rangle \mapsto |0\rangle$, $|1\rangle \mapsto -|1\rangle$ 'phase shift'

$\sigma_y = \begin{bmatrix}0&-i\\i&0\end{bmatrix}: \binom{\alpha}{\beta} = \binom{-i\beta}{i\alpha} = -i\binom{\beta}{-\alpha}$

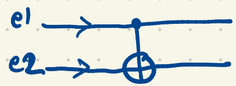


$$\begin{aligned}|00\rangle &\mapsto |10\rangle\\ |01\rangle &\mapsto |11\rangle\\ |10\rangle &\mapsto |00\rangle\\ |11\rangle &\mapsto |01\rangle\end{aligned}$$

$$\left[\begin{array}{cc|cc}0&1&0&0\\1&0&0&0\\\hline 0&0&0&1\\0&0&1&0\end{array}\right] = \sigma_x \otimes \begin{bmatrix}1&0\\0&1\end{bmatrix}$$

wrt basis $|00\rangle, |10\rangle, |01\rangle, |11\rangle$

CNOT gate: $|00\rangle \mapsto |00\rangle$, $|01\rangle \mapsto |01\rangle$, $|10\rangle \mapsto |11\rangle$, $|11\rangle \mapsto |10\rangle$



$$\left[\begin{array}{cc|cc}1&&&\\&1&&\\\hline &&&1\\&&1&\end{array}\right]$$

wrt basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$

In quantum computation, quantum information is often modeled as qubits.
An ensemble of n electrons has spin state $|\psi\rangle \in \underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n} \cong \mathbb{C}^{2^n}$ (unit vector).

Can initialize $|\psi\rangle$ in a particular state, usually $|000\cdots 0\rangle = |0\rangle \otimes \cdots \otimes |0\rangle$ (all electrons spin up)

Cannot clone a qubit. Measurement of a qubit yields at most n classical bits of information.

Can perform reversible processes $|\psi\rangle \longmapsto U|\psi\rangle$, U unitary $2^n \times 2^n$ matrix.

Can measure $|\psi\rangle$, typically by measuring spin of each electron separately.

Kochen-Specker Theorem 1967, as simplified by Peres shortly after.

Electrons vs. Photons

Spin $\in \mathbb{C}^2$ (unit vector)
Measurement yields one classical bit $+$ or $-$ (up : down)

Spin $\in \mathbb{C}^3$
Measurement wrt orthonormal frame (x, y, z)
Spin (when measured) gives $0,1,1$ or $1,0,1$ or $1,1,0$
classical trit (ternary digit)

We designate 33 choices of axis in $\mathbb{R}^3$ prescribed by the Peres configuration containing 24 different orthogonal frames (an orthogonal frame is a choice of (x,y,z) coordinate system with perpendicular axes).

Take an EPR pair of photons (photons 1 and 2) separated by a vast distance.

Spin is actually measured in units of $\frac{\hbar}{2}$, $\hbar = h/2\pi$

Electron spin: $-\frac{\hbar}{2}$ or $\frac{\hbar}{2}$ spin $\frac{1}{2}$ particle (fermion)

Photon spin: $-\hbar$, 0, $\hbar$ spin 1 particle (boson)

CHSH game: Alice and Bob.

when $(x,y) = \{(0,0), (1,0), (0,1)\}$, Alice and Bob win the round when their guesses agree

when $(x,y) = (1,1)$, Alice and Bob win the round when their guesses disagree.

Strategy for winning 75% of the time: Both Alice and Bob guess '+' every round.

Can't beat this (apparently) without EPR pairs.

Consider an electron pair e1, e2 in spin state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}|0\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}}|1\rangle \otimes |1\rangle$

Alice has e1, Bob has e2. Alice chooses a pair of states $|+_A\rangle, |-_A\rangle$ which is an orthonormal basis for $\mathbb{C}^2$. Bob has $|+_B\rangle, |-_B\rangle$.

$\binom{\alpha}{\beta}$ $\binom{\bar{\beta}}{\bar{\alpha}}$ $\underbrace{|\alpha|^2 + |\beta|^2 = 1}_{\langle +_A|+_A\rangle = 1}$, $|\gamma|^2 + |\delta|^2 = 1 = \langle -_A|-_A\rangle$

$\underbrace{\bar{\alpha}\gamma + \bar{\beta}\delta = 0}_{\langle +_A|-_A\rangle = 0}$

Probability that Alice and Bob agree in their spin measurement is $|\langle +_A|+_B\rangle|^2$.

Born's Rule: If Alice measures an electron in state $|\psi\rangle \in \mathbb{C}^2$ with respect to her choice of basis, the prob. of spin up is $|\langle +_A|\psi\rangle|^2$

- - - - - - - - - down is $|\langle -_A|\psi\rangle|^2$ So $|\langle +_A|\phi\rangle|^2 + |\langle -_A|\psi\rangle|^2 = 1.$ $\leftarrow$ $\||\psi\rangle\|^2 = \langle\psi|\psi\rangle$

In particular $|\langle +_A|+_B\rangle|^2 = 1 - |\langle -_A|+_B\rangle|^2 = |\langle -_A|-_B\rangle|^2$

If Alice and Bob measure their electrons, what is the probability they agree on spin direction?

Assume Alice measures first:

- Suppose Alice measures e1 to be spin up. This occurs with probability $|\langle +_A|\psi\rangle|^2$.
  In this case e1 collapses into state $|+_A\rangle$. Instantly we know e2 is also in this state $|+_A\rangle$. The prob. that Bob also measures e2 to be 'up' is $|\langle +_B|+_A\rangle|^2$.
  The prob. that all this occurs is $|\langle +_A|\phi\rangle|^2 |\langle +_B|+_A\rangle|^2$.
- Suppose Alice measures e1 to be spin down. Prob. (this) $= |\langle -_A|\psi\rangle|^2$.
  In this case e1 and e2 are in state $|-_A\rangle$. The prob. that Bob also finds e2 to be in 'down' state is $|\langle -_B|-_A\rangle|^2$. All this occurs with prob. $|\langle -_A|\phi\rangle|^2 |\langle -_B|-_A\rangle|^2$.

Total prob. that Alice and Bob 'agree' is

$$|\langle +_A|\phi\rangle|^2 |\langle +_A|+_B\rangle|^2 + |\langle -_A|\psi\rangle|^2 |\langle +_A|+_B\rangle|^2 = \underbrace{\left(|\langle +_A|\phi\rangle|^2 + |\langle -_A|\psi\rangle|^2\right)}_{1} |\langle +_A|+_B\rangle|^2 = |\langle +_A|+_B\rangle|^2!$$

Alice and Bob can win 85.4% of the time at the CHSH game using EPR pairs.

Say each EPR pair is in spin state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ e1, e2

Alice uses $|+_{A_0}\rangle = \begin{bmatrix}1\\0\end{bmatrix}$, $|-_{A_0}\rangle = \begin{bmatrix}0\\1\end{bmatrix}$ if the referee sends her $x=0$

" " $|+_{A_1}\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$, $|-_{A_1}\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$ " — — — — — " $x=1$

Bob use $|+_{B_0}\rangle = \frac{1}{\sqrt{4+2\sqrt{2}}}\begin{bmatrix}-1\\1+\sqrt{2}\end{bmatrix}$, $|-_{B_0}\rangle = \frac{1}{\sqrt{4+2\sqrt{2}}}\begin{bmatrix}1+\sqrt{2}\\1\end{bmatrix}$ if the ref sends him $y=0$

" " $|+_{B_1}\rangle = \frac{1}{\sqrt{4+2\sqrt{2}}}\begin{bmatrix}1\\1+\sqrt{2}\end{bmatrix}$, $|-_{B_1}\rangle = \frac{1}{\sqrt{4+2\sqrt{2}}}\begin{bmatrix}1+\sqrt{2}\\-1\end{bmatrix}$ " — — — — " " $y=1$.

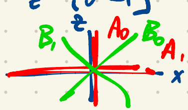
If Alice and Bob measure their respective electrons e1, e2 with their choice of bases A_x, B_y then the probability that they "agree"

$$|\langle +_{A_x} | +_{B_y}\rangle|^2 = \begin{cases} \frac{2+\sqrt{2}}{4} \approx 0.854 & \text{if } (x,y) \in \{(0,0),(1,0),(0,1)\} \\ 1 - \frac{2+\sqrt{2}}{4} & \text{if } (x,y) = (1,1). \end{cases}$$

In all cases $(x,y) \in \{0,1\}^2$ Alice and Bob have a probability $\frac{2+\sqrt{2}}{4} \approx 0.854$ of winning their round at CHSH game.

Bell's Theorem: Under certain reasonable assumption, hidden variable theories give at most 75% chance of Alice and Bob winning at the CHSH game.

A_0 : $\sigma_z = \begin{bmatrix}1&0\\0&-1\end{bmatrix}$, A_1: $\sigma_x = \begin{bmatrix}0&1\\1&0\end{bmatrix}$, $n = (n_x, n_y, n_z) \in \mathbb{R}^3$ $n \cdot (\sigma_x, \sigma_y, \sigma_z)$ B_0, B_1



"Superdense coding" or "dense coding"

A way to store and retrieve 2 classical bits in one qubit. (Using EPR pairs)

Alice and Bob are far apart. They have a shared EPR pair $e1, e2$ in state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

At some later time, Alice wants to send a pair of classical bits 00, 01, 10 or 11 to Bob.

Alice performs a reversible operation U (2×2 unitary matrix) on her electron $e1$.

She then sends $e1$ to Bob. (no faster than the speed of light)

Bob has $e1$ now as well as $e2$. He performs a measurement and retrieves Alice's bit pair

Alice has two classical bits, one of the four cases:

- 00, Alice applies $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ to $e1$. (She does nothing to it.)
- 01, Alice applies $\sigma_x = \begin{bmatrix}0&1\\1&0\end{bmatrix}$ to $e1$, mapping $|\psi\rangle$ to B_{01}.
- 10, Alice applies $\sigma_z = \begin{bmatrix}1&0\\0&-1\end{bmatrix}$ to $e1$, ... to B_{10}
- 11, Alice applies $\sigma_z\sigma_x = \begin{bmatrix}0&1\\-1&0\end{bmatrix} = i\sigma_y$ to $e1$, mapping $|\psi\rangle$ to B_{11}.

$|00\rangle, |01\rangle, |10\rangle, |11\rangle$ is an orthonormal basis of $\mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2$

Bell basis is another such basis:

$$B_{00} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$
$$B_{01} = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$
$$B_{10} = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$
$$B_{11} = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Alice sends the transformed $e1$ to Bob.

Bob reversibly applies a 4×4 unitary matrix mapping

$$B_{00} \mapsto |00\rangle$$
$$B_{01} \mapsto |01\rangle$$
$$B_{10} \mapsto |10\rangle$$
$$B_{11} \mapsto |11\rangle$$

then he measures $e1, e2$ and thereby recovers Alice's pair of classical bits.