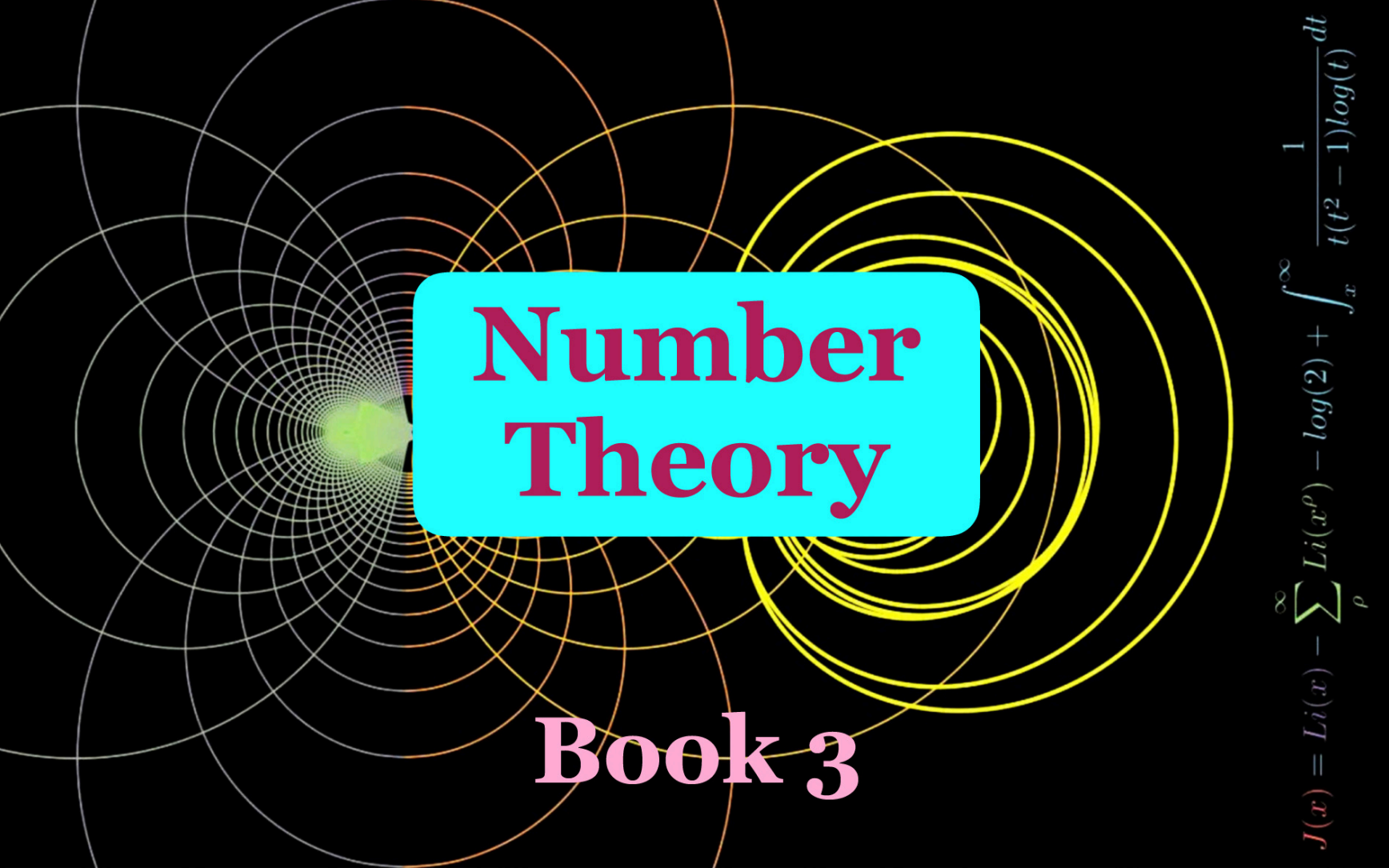

Number
Theory
Book 3
$J(x) = Li(x) - \sum_{\rho}^{\infty} Li(x^{\rho}) - log(2) + \int_{x}^{\infty} \frac{1}{t(t^2-1)log(t)} dt$

An elliptic curve $E(\mathbb{F}_q)$ over a finite field is an additive abelian group which is either cyclic C_n or a direct sum $C_n \oplus C_m$ (typically $n \mid m$ with n small).
The order $|E(\mathbb{F}_q)| \in [q+1-2\sqrt{q},\ q+1+2\sqrt{q}]$ by the Hasse bound or HW bound.

$x^2+y^2=-1$ is a curve of genus 0 (nonsingular conic)
Over $\mathbb{F}_q$ (q odd) we have $q+1$ points.
For curves of genus $g \geq 2$: $X(\mathbb{Q})$ is a finite set. Mordell's Conjecture, proved by Gerd Faltings (1983).
(set of rational points)
eg. for any fixed $n \geq 3$, the equation $x^n + y^n = 1$ has at most finitely many rational points.
(For $n=3$, $g=1$, elliptic curve, Fermat curve curve with finitely many rational points.
For $n \geq 4$, $g > 1$.) This precedes the proof by Wiles & others of Fermat's Last Thm.

Elliptic curves have applications in cryptography and primality testing and integer factorization.
AKS: There is a deterministic poly.-time algorithm for deciding whether or not a given integer is a prime. Given n, the running time is bounded by $c \cdot (\log n)^b$.
For practical implementation, however, we almost always still use probabilistic algorithms.
Fermat test: Given n, pick $a \in \{2, \dots, n-1\}$ randomly. Compute $a^{n-1} \bmod n$.
If $a^{n-1} \not\equiv 1 \bmod n$, return "$n$ is composite."
If $a^{n-1} \equiv 1 \bmod n$, pick a different $a \in \{2, \dots, n-1\}$ and repeat.
If we stop after 100 trials (say) then we have a one-sided error ("false positive").

Rabin-Miller: improvement of Fermat test. Less likelihood of error but it still has one-sided errors (false positives).

The Elliptic curve test for primality: given n, we do a certain computation.
If n passes the test then n is guaranteed to be prime
If n fails the test, repeat the test with a different point on a different curve.
One-sided error (false negative) if we decide to stop after 100 trials, say.
Combining Rabin-Miller with Elliptic curve test (alternately) it is extremely unlikely to not get a guarantee in 100 trials.

Example using Elliptic Curves to prove primality of $n = 10^{10} + 19 = 10{,}000{,}000{,}019$
using the Goldwasser-Kilian algorithm (1986?) believed to be poly-time (probabilistic).
We take an elliptic curve over $\mathbb{Z}/n\mathbb{Z}$ chosen randomly but with a known point
$P \in E(\mathbb{Z}/n\mathbb{Z})$.
What can we do with $F = \mathbb{Z}/n\mathbb{Z}$? If n = prime then F is a field.
If $n = pq$, $p \neq q$ large primes (say hundreds of digits) then F is not a field.
Imagine $n = pq$ composite but very difficult to factor if p, q large.
Given $a, b \in F$. $\frac{a}{b}$ is practically defined if $b \neq 0$. $\gcd(b, n) \in \{1, p, q\}$
The case $\gcd(b, n) \in \{p, q\}$ only arises if we are able to factor n, so we return "n is not a prime". Otherwise $\gcd(b, n) = 1 = rb + sn$ for some $r, s \in \mathbb{Z}$ by Euclid's Algorithm
so $\frac{1}{b} = r$ in F. The computation proceeds.

If n is not prime and $n = kq$, $k > 1$, q prime,

$E(\mathbb{Z}/n\mathbb{Z}) \xrightarrow{\text{almost-homo}} E(\mathbb{Z}/q\mathbb{Z})$

"elliptic curve over $\mathbb{Z}/n\mathbb{Z}$" but not strictly speaking an elliptic curve since $\mathbb{Z}/n\mathbb{Z}$ is not a field.

actual elliptic curve

meaning: if $P, Q \in E(\mathbb{Z}/n\mathbb{Z})$ where $P+Q \in E(\mathbb{Z}/n\mathbb{Z})$ is well-defined

then $\overline{P} + \overline{Q} = \overline{P+Q}$ where "bar" is "mod q".

Example Show $n = 10^{10}+19 = 10000000019$ is prime. Proof by contradiction.

Supposing n is composite n has a prime factor $p \le \sqrt{n}$ so $p < 10^5 = 100{,}000$.

This will lead to a contradiction.

Randomly I choose elliptic curves $y^2 = x^3 + ax + b$ containing a point $P = (3, 5)$.

To check that this is an elliptic curve, need $4a^3 + 27b^2 \not\equiv 0 \bmod n$ (discriminant of $x^3 + ax + b$)

E	$\lvert E(\mathbb{Z}/n\mathbb{Z})\rvert$
$y^2 = x^3 + x - 5$	$9999935488 = 2^6 \cdot 3 \cdot 19 \cdot 4761 \cdot 7649$
$y^2 = x^3 + 2x - 8$	$10000162104 = 2^3 \cdot 3 \cdot 29 \cdot 547 \cdot 2687$
$y^2 = x^3 + 3x - 11$	$10000053492 = 2^2 \cdot 3 \cdot 11 \cdot 13 \cdot 37 \cdot 239 \cdot 659$
$y^2 = x^3 + 4x - 14$	$9999892527 = 3 \cdot 19 \cdot 89 \cdot 1971199 = m$ (k, q)

Corrected! $4a^3 + 27b^2 = (27b - 18ax)(x^3 + ax + b) + (4a^2 - 9bx + 6ax^2)(3x^2 + a)$

To prove that n is prime, we argue by contradiction. If not there is a prime $p \le \sqrt{n}$, $p \mid n$.

So $p < 100{,}000$. Then there is an almost-homomorphism $E(\mathbb{Z}/n\mathbb{Z}) \to E(\mathbb{F}_p)$

So $E(\mathbb{F}_p)$ has a point of order q

Check: $mP = 0$, $kP \neq 0$.

The point $kP \in E(\mathbb{Z}/n\mathbb{Z})$ has order q.

We have presumably checked recursively that q is prime.

"elliptic curve" over $\mathbb{Z}/n\mathbb{Z}$

actual elliptic curve over $\mathbb{F}_p$.

$E(\mathbb{F}_p)$ has order $\leq p+1+2\sqrt{p} \leq 100,632$ but q is bigger than this, contradiction.

A revised version of Goldwasser-Kilian algorithm replaces the arbitrary elliptic curves with special curves called CM curves where the group order is easier to compute.

Elliptic Curve Factorization Method (Lenstra)

Given n large integer which is known to be composite, we want to split $n = ab$, $1 < a, b < n$ (nontrivial factorization).

Choose random elliptic curves $E(\mathbb{Z}/n\mathbb{Z})$ with known point P.

Do extensive sums in $\langle P \rangle = \{kP : k \in \mathbb{Z}\}$ until we find a failure of chord-tangent method where division by $b \in \mathbb{Z}/n\mathbb{Z}$ fails, $\gcd(b, n) \in \{2, \ldots, n-1\}$ giving a splitting of n.

This is subexponential time in practice, like the best sieve methods.

Applications to public key cryptography:

Classical Diffie-Hellman protocol for key distribution: This allows two parties to agree on a secret key (typically long alphanumeric string) while communicating over an insecure channel.

We want to generate a secret key over an open channel: a secret number which should be hundreds of digits long. How to do this:

Alice and Bob first choose a large prime p (probably a few hundred digits long). They also agree on a primitive element g mod p i.e. all nonzero elements of $\mathbb{F}_p$ are powers of g. (Eg. if $p = 1009$, $g = 11$ is primitive. In $\mathbb{F}_{1009}$

$11^0 = 1$
$11^1 = 11$
$11^2 = 121$
$11^3 = 1331 = 322$
$11^4 = 515$
$11^5 = 620$
$\vdots$
$11^{1007} = 367$
$11^{1008} = 1$
$11^{1009} = 11$
$\vdots$

where is $186 \in \mathbb{F}_{1009}$ in this list?

$11^k = 186$ in $\mathbb{F}_{1009}$ for some unique $k \in \{0, 1, 2, \cdots, 1007\}$.

What is k? $k = 543$ is the answer.

Why is the set of nonzero elements of a finite field a cyclic group?
eg. $\mathbb{F}_5$ has four nonzero elements forming a multiplicative group of order 4. If this group is a Klein 4-group then the poly. $x^2 - 1$ has four roots in $\mathbb{F}_5$.

Alice and Bob agree on p and g (as above) over the open channel (not secure).
Secretly, Alice chooses $a \in \{1, 2, \cdots, p-2\}$ at random. She computes $g^a \in \mathbb{F}_p$ and sends this to Bob (over the open channel).
Bob (secretly) chooses $b \in \{1, 2, \cdots, p-2\}$ and computes $g^b \in \mathbb{F}_p$ and sends this to Alice.
The information p, g, g^a, g^b have been shared over the open channel; a, b are secret.

The secret key known to Alice and Bob is g^{ab}.

Alice computes $(g^b)^a = g^{ab}$

Bob computes $(g^a)^b = g^{ab}$

Is there a shortcut to computing g^{ab} without first finding a or b?

Not as far as we know.

ElGamal uses Diffie-Hellman

Other groups in place of $\mathbb{F}_q^\times = \{a \in \mathbb{F}_q : a \neq 0\}$

However elliptic curves over finite field can provide the same amount of security with shorter key length

Substitute the group of the curve for $\mathbb{F}_q^\times$

Fix curve, P point.

Alice chooses large integer a, computes aP, sends this to Bob.

Bob - - - - - - b, computes bP, - - - - Alice.

The secret key is $abP = a(bP) = b(aP)$.

Modular Forms (and Elliptic Curves) - Neal Koblitz

Example: the j-invariant, $j(\tau)$ is usually expressed as a function of $q = e^{2\pi i \tau}$

First define $g_2(\tau) = 60 \sum_{(0,0)\neq(m,n)\in\mathbb{Z}^2} \frac{1}{(m+n\tau)^4}$

$g_3(\tau) = 140 \sum_{(0,0)\neq(m,n)\in\mathbb{Z}^2} \frac{1}{(m+n\tau)^6}$

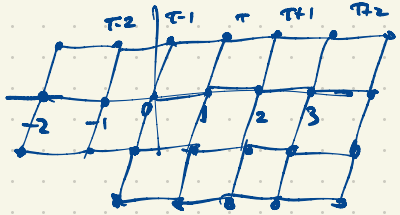
$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n)$

$$j(\tau) = 1728 \frac{g_2(\tau)^3}{g_2(\tau)^3 - 27 g_3(\tau)^2} = \frac{1728\, g_2(\tau)^3}{(2\pi)^{12}\, \eta(\tau)^{24}}$$

$12^3 = 1728$

$$= \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + \cdots$$

$g_2(\tau)$, $g_3(\tau)$ depend only on the lattice $\{m+n\tau : m,n\in\mathbb{Z}\} \subset \mathbb{C}$ so $j(\tau)$ only depends on this lattice



The Monster (largest sporadic finite simple group) $M = FG$ can be written as a subgroup (isomorphic to) of $GL_{196883}(\mathbb{C})$

Monstrous Moonshine

$\mathbb{C}/\{m+n\tau : m,n\in\mathbb{Z}\} \cong$ torus $T^2 \cong S^2 \cong S^1 \times S^1$

$\cong$ the elliptic curve $y^2 = 4x^3 - g_2(\tau)x - g_3(\tau)$.

Two such curves are equivalent up to analytic isomorphism iff they have the same j-invariant.

Scaling a lattice $L \subset \mathbb{C}$ by $\alpha \neq 0$ ($\alpha \in \mathbb{C}$) gives αL with $\mathbb{C}/\alpha L \simeq \mathbb{C}/L$

elliptic curves are essentially the same.

$L = \mathbb{Z}u + \mathbb{Z}v$ (u, v base for L)
$= \{au + bv : a, b \in \mathbb{Z}\}$

WLOG $u = 1$ otherwise scale the entire lattice by u^{-1}.

Then $L = \mathbb{Z} + \mathbb{Z}\tau = \langle 1, \tau \rangle$

$\tau \to 1$

Also we may assume τ is in the upper half-plane ($\operatorname{Im}\tau > 0$) otherwise pick new basis.

$\{1, 1+\tau\}$ generates the same lattice hence the same elliptic curve $\mathbb{C}/\langle 1, \tau \rangle$

Also $\langle 1, \tau^{-1} \rangle$ gives essentially the same curve

↓ scale by τ

$\langle \tau, 1 \rangle$

More generally, the group $SL_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} = g : a, b, c, d \in \mathbb{Z}, \ ad - bc = 1 \right\}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

acts on $\mathbb{C} \cup \{\infty\}$ by fractional linear transformations $g(z) = \frac{az+b}{cz+d}$, $g^{-1}(z) = \frac{dz-b}{-cz+a}$

The map $\tau \mapsto \tau + 1$ is really $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}(\tau) = \frac{1\tau + 1}{0\tau + 1} = \tau + 1$

$\tau \mapsto -\tau^{-1}$ . . . . $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}(\tau) = \frac{0\tau + 1}{-\tau + 0} = -\frac{1}{\tau}$.

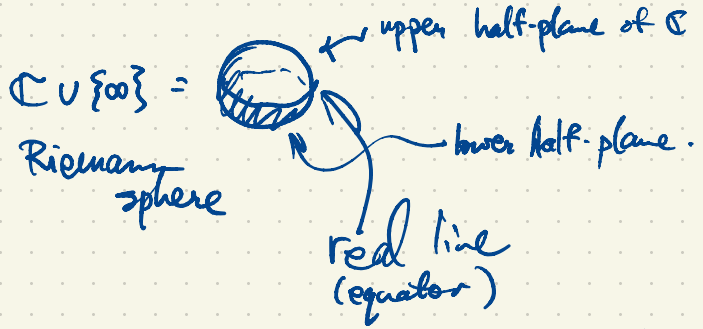
But $\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle = SL_2(\mathbb{Z})$ (generate as group)

Actually $SL_2(\mathbb{Z}) \to PSL_2(\mathbb{Z})$ is 2-to-1 homomorphism

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \begin{bmatrix} a & -b \\ -c & -d \end{bmatrix}$

$g, -g \in SL_2(\mathbb{Z})$

give the same fractional linear transformation



$SL_2(\mathbb{Z})$ maps $\mathbb{R}\cup\{\infty\} \longrightarrow \mathbb{R}\cup\{\infty\}$ (equator)

upper half-plane $\longrightarrow$ upper half-plane

lower $\cdots\longrightarrow$ lower half-plane

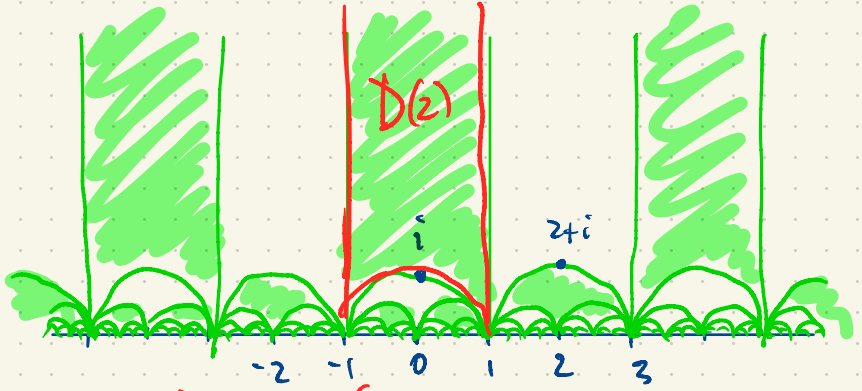
$PSL_2(\mathbb{Z})$ has no subgroup of index 2 (using a simplicity argument)

$j(\tau) = j(\tau+1) = j(\tau^{-1})$ so $j\left(\frac{a\tau+b}{c\tau+d}\right) = j(\tau)$ for all $a,b,c,d\in\mathbb{Z}$, $ad-bc=1$.

$j(\tau)$ is invariant under the modular group $PSL_2(\mathbb{Z})$.

We construct a fundamental domain $D \subset H = \{\tau\in\mathbb{C} : \mathrm{Im}\,\tau > 0\}$ upper half-plane:

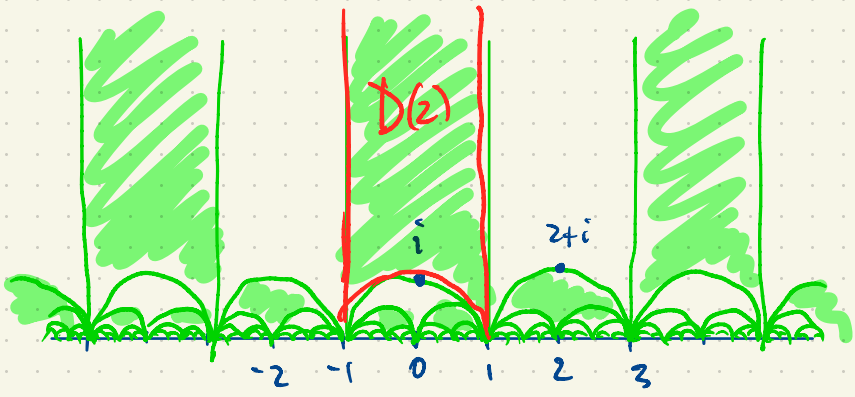
For every point in H there is a unique $g\in PSL_2(\mathbb{Z})$ mapping it into D

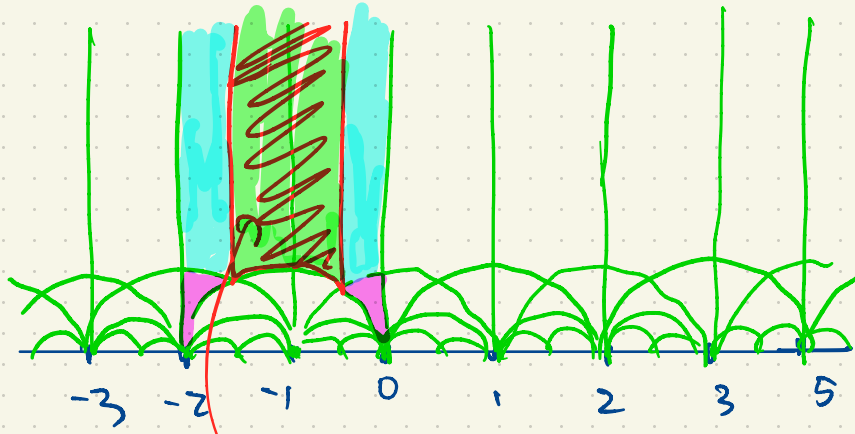


D(2) is a fund. domain for $\Gamma(2) < \Gamma$ generated by $\tau\mapsto\tau+2$, $\tau\mapsto -\frac{1}{\tau}$

$\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\in\Gamma(2)$ $\quad$ $\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\in\Gamma(2)$

$$D(2) = \{\tau\in\mathbb{C} : \mathrm{Im}\,\tau > 0,\ -1 < \mathrm{Re}\,\tau < 1,\ |\tau| > 1\}$$





$$D = \{\tau \in \mathbb{C} : \operatorname{Im}\tau > 0,\ -\tfrac{1}{2} < \operatorname{Re}\tau < \tfrac{1}{2},\ |\tau| > 1\}$$

$PSL_2(\mathbb{Z}) = \{$ fractional linear transformations $g(z) = \frac{az+b}{cz+d} : a,b,c,d \in \mathbb{Z},\ ad-bc=1\}$

$= SL_2(\mathbb{Z})/\{\pm I\} = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{Z},\ ad-bc=1\} / \underbrace{\{\pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\}}_{Z = Z(SL_2(\mathbb{Z}))}$

$\Gamma(p) = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bmod p\}$, p prime (principal congruence subgp)

$a \equiv d \equiv 1 \bmod p$, $c \equiv d \equiv 0 \bmod p$

normal subgp of $SL_2(\mathbb{Z})$

$PSL_2(\mathbb{Z})/\Gamma(p) \cong PSL_2(\mathbb{F}_p)$

$PSL_2(\mathbb{Z}) \xrightarrow{\text{onto}} PSL_2(\mathbb{F}_p)$ by taking all entries mod p; its kernels is $\Gamma(p)$.

simple group for $p > 3$.

$\Gamma(2) = \begin{bmatrix} O & E \\ E & O \end{bmatrix} = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL_2(\mathbb{Z}) : a,d \text{ odd};\ b,c \text{ even}\} =$ "identity mod 2"

$\Gamma = \begin{bmatrix} O & E \\ E & O \end{bmatrix} \sqcup \begin{bmatrix} E & O \\ O & E \end{bmatrix} \sqcup \begin{bmatrix} O & O \\ E & O \end{bmatrix} \sqcup \begin{bmatrix} O & E \\ O & O \end{bmatrix} \sqcup \begin{bmatrix} O & O \\ O & E \end{bmatrix} \sqcup \begin{bmatrix} E & O \\ O & O \end{bmatrix}$

$PSL_2(\mathbb{F}_2) = \{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\} \cong SL_2(\mathbb{F}_2) \cong GL_2(\mathbb{F}_2) \cong PGL_2(\mathbb{F}_2)$

$[\Gamma : \Gamma(2)] = 6$ so a fundamental domain for $\Gamma(2)$ has six copies of D (fund. domain for Γ).

Counting representations of n as a sum of squares:

$$\theta(q) = \sum_{k\in\mathbb{Z}} q^{k^2} = \cdots + q^9 + q^4 + q + 1 + q + q^4 + q^9 + \cdots = 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + \cdots$$

$\cdots\ k=-3 \quad k=-2 \quad k=-1 \quad k=0 \quad k=1 \quad k=2 \quad k=3 \ \cdots$

Generating function for the number of ways of writing n as a sum of two squares:

$$\theta(q)^2 = 1 + 4q + 4q^2 + 4q^4 + 8q^5 + 4q^8 + 4q^9 + 8q^{10} + \cdots$$

$$= (1 + 2q + 2q^4 + 2q^9 + \cdots)(1 + 2q + 2q^4 + 2q^9 + \cdots)$$

$5 = a^2 + b^2$ for $(a,b) \in \{(\pm 1, \pm 2), (\pm 2, \pm 1)\}$

$$\theta(q)^4 = 1 + 8q + 24q^2 + 32q^3 + 24q^4 + \cdots = \sum_{n=0}^{\infty} (\quad\quad) q^n$$

$(\pm 1, 0, 0, 0)$, $(0, \pm 1, 0, 0)$, $\cdots$
$(\pm 1, \pm 1, 0, 0)$, $\cdots$
$(\pm 1, \pm 1, \pm 1, 0)$
$(\pm 1, \pm 1, \pm 1, \pm 1)$, $(\pm 2, 0, 0, 0)$

For $n > 0$, $8\sigma(n) - 32\sigma(\frac{n}{4})$

$\sigma(1) = 1$
$\sigma(2) = 1+2 = 3$
$\sigma(3) = 1+3 = 4$
$\sigma(4) = 1+2+4 = 7$

$8 \cdot 7 - 32 \cdot 1 = 56 - 32 = 24$

$\sigma(n)$ = Sum of pos. divisors of n (if n pos. int.)

$\sigma(mn) = \sigma(m)\sigma(n)$ whenever $\gcd(m,n) = 1$

$\theta(q)^n$ are examples of modular functions.
(not for the full modular group Γ but for certain of its subgroups)

A lattice $L \subset \mathbb{R}^n$ is the set of $\mathbb{Z}$-linear combinations of a basis $v_1, \dots, v_n$.
L is integral if $x \cdot y \in \mathbb{Z}$ for all $x, y \in L$.

eg. in $\mathbb{R}^8$, $\mathbb{Z}^8 = \{(a_1, \dots, a_8) : a_i \in \mathbb{Z}\} \subset \mathbb{R}^8$ is integral

$E_8 = \{\frac{1}{2}(a_1, \dots, a_8) : a_i \in \mathbb{Z},\ a_1 \equiv \cdots \equiv a_8 \bmod 2,\ \sum a_i \equiv 0 \bmod 4\}$ is also integral.

Moreover both $\mathbb{Z}^8$ and E_8 are self-dual:

Given a lattice $L \in \mathbb{R}^n$, the dual lattice $L^* \subset \mathbb{R}^n$ is $L^* = \{v \in \mathbb{R}^n : v \cdot L \in \mathbb{Z}\}$
$= \{v \in \mathbb{R} : v \cdot x \in \mathbb{Z} \text{ for all } x \in L\}$

L an integral $\Rightarrow L \subseteq L^*$
L is self-dual iff $L = L^*$
iff the $v_1, \dots, v_n$ form the rows of an $n \times n$ matrix of determinant ± 1.

Assume L is an integral lattice. The theta-series of L is $(q = e^{\pi i \tau})$

$$\Theta_L(\tau) = \sum_{v \in L} q^{v \cdot v} = \sum_{n=0}^{\infty} N_L(n) q^n \quad \text{where } N_L(n) = \text{number of vectors } v \in L \text{ of norm } n.$$

eg. $\Theta_{\mathbb{Z}}(\tau) = 1 + 2q + 2q^4 + 2q^9 + \cdots = \theta(\tau)$ (or use q as argument)

$\Theta_{\mathbb{Z}^8}(\tau) = \theta(\tau)^8 = 1 + 16q + 112q^2 + \cdots$

16 vectors of norm 1: $(\pm 1, 0, \dots, 0), \dots$
Vectors of norm 2: $(\pm 1, \pm 1, 0, \dots, 0), \dots$
$4 \cdot \binom{8}{2} = 4 \cdot \frac{8 \cdot 7}{2} = 112$ such vectors

$\Theta_{E_8}(\tau) = 1 + 240q^2 + 2160q^4 + \cdots$
$= 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^{2n}$

All $v \in E_8$ have even norm

where $\sigma_k(n) =$ sum of k^{th} powers of the positive integer divisors of n.

$\sigma_1(n) = \sigma(n)$

$\sigma_k(mn) = \sigma_k(m)\sigma_k(n)$ when m, n rel. prime.

$\sigma_3(1) = 1^3 = 1$
$\sigma_3(2) = 1^3 + 2^3 = 9$, $\quad 9 \cdot 240 = 2160$

$E_8 = \{ \frac{1}{2}(a_1, \dots, a_8) : a_i \in \mathbb{Z},\ a_1 \equiv \cdots \equiv a_8 \bmod 2,\ \sum a_i \equiv 0 \bmod 4 \}$ is also integral.

The shortest nonzero vectors in E_8 are the 240 root vectors

$\frac{1}{2}(\pm 2, \pm 2, 0, \dots, 0) = (\pm 1, \pm 1, 0, \dots, 0)$ 112 such

$\frac{1}{2}(\pm 1, \pm 1, \pm 1, \dots, \pm 1)$ (even number of $+1$'s), 128 such

} 240 root vectors

If L is any integral lattice then Θ_L and Θ_{L^*} are related by the Jacobi theta-function identity

$$\Theta_{L^*}(\tau) = \sqrt{|\det L|}\ \left(\frac{i}{\tau}\right)^{n/2} \Theta_L\left(-\frac{1}{\tau}\right)$$

For self-dual (integral) lattices in $\mathbb{R}^8$ ($n = 8$), $\Theta_L(\tau) = \tau^4\, \Theta_L\left(-\frac{1}{\tau}\right)$

i.e. $\Theta_L(\tau)$ is a modular form of weight 4 for the subgroup of index 3 in $\Gamma = PSL_2(\mathbb{Z})$ namely the subgroup $G(2) = \{ z \mapsto \frac{az+b}{cz+d} : a, b, c, d \in \mathbb{Z},\ ad - bc = 1,\ \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\begin{bmatrix} 0 & E \\ E & 0 \end{bmatrix}$ or $\begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} \}$

$[\Gamma : G(2)] = 3$. $\quad \tau \mapsto -\frac{1}{\tau}$ is in $G(2)$

$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}(\tau) = \frac{0\tau + 1}{-\tau + 0}$

$G(2)$ is generated by $\tau \mapsto -\frac{1}{\tau}$ and $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}(\tau) = \frac{1\tau + 2}{0\tau + 1} = \tau + 2$

$\tau \mapsto \tau + 2$ which satisfies $\Theta_L(\tau + 2) = \Theta_L(\tau)$

More generally, a <u>modular form</u> of weight k for a subgroup $G \leq \Gamma = PSL_2(\mathbb{Z})$ is a $\mathbb{C}$-valued function $f(z)$ defined and holomorphic (complex analytic) in the upper half-plane $\{ z \in \mathbb{C} : \operatorname{Im} z > 0 \}$ such that

(i) $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ for every $z \mapsto \frac{az+b}{cz+d}$ in G; and

(ii) the Laurent expansion of $f(z)$ in $q = e^{\pi i z}$ has no negative powers of q i.e. $f(\tau) = \sum_{n=0}^{\infty} a_n q^n$.

Restricting to a subgroup $G \leq \Gamma$ of (small) finite index, the set of modular forms of weight k for G is a finite dimensional vector space over $\mathbb{C}$. The dimension for certain G, k have been computed (Ogg et al). Eg. for the group $G = G(2) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{Z},\ ad-bc=1,\ a \equiv d \bmod 2,\ b \equiv c \bmod 2 \right\}$, $[\Gamma : G] = 3$, modular forms of weight 4 for G satisfy

$$f(z+2) = f(z), \qquad f\left(-\tfrac{1}{z}\right) = z^4 f(z)$$

(2 is a period)

and the modular forms form a 2-dimensional vector space.
Two examples of modular forms are known:

$$\left.\begin{aligned} \textcircled{H}_{E_8}(\tau) &= 1 + 240q^2 + 2160q^4 + \cdots \\ \textcircled{H}_{\mathbb{Z}^8}(\tau) &= 1 + 16q + 112q^2 + \cdots \end{aligned}\right\} \text{ basis}$$

We also know that $g_2(\tau) = 60 \sum_{(0,0) \neq (m,n) \in \mathbb{Z}^2} \frac{1}{(m+n\tau)^4}$ is a modular form for G.

$$g_2(\tau+2) = 60 \sum_{m,n} \frac{1}{(m+n(\tau+2))^4} = 60 \sum_{m,n} \frac{1}{(m+n\tau)^4} = g_2(\tau) \text{ by change of dummy variable.}$$

$$g_2\left(-\tfrac{1}{\tau}\right) = 60 \sum_{m,n} \frac{1}{\left(m - \frac{n}{\tau}\right)^4} = 60 \sum_{m,n} \frac{\tau^4}{(m\tau - n)^4} = \tau^4 g_2(\tau)$$

Actually the Laurent expansion $g_2(z) = \boxed{2\zeta(4)}\left[1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^{2n}\right]$ is not hard to prove ← not quite

$$\zeta(2) = \frac{\pi^2}{6} \qquad \zeta(4) = \frac{\pi^4}{90}, \qquad \zeta(2k) = \frac{(-1)^{k+1}(2\pi)^{2k} B_{2k}}{2(2k)!}$$ ← Bernoulli numbers

$\frac{1}{2\zeta(4)} g_2(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^{2n}$ is a $\mathbb{C}$-linear combination of $\textcircled{H}_{E_8}(\tau)$ and $\textcircled{H}_{\mathbb{Z}^8}(\tau)$. Solve 2 eqns in 2 unknowns to get

$$\textcircled{H}_{E_8}(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^{2n}.$$ Can we prove this without using any analysis? Probably but... try!

Can you find two lattices $L, L' \subset \mathbb{R}^n$ with the same theta series?

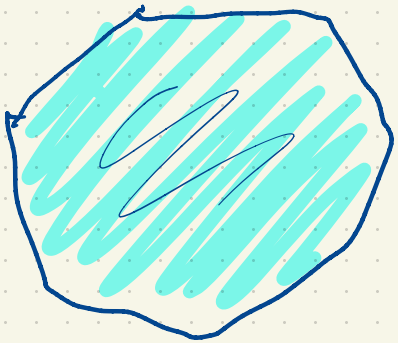
Theta series Θ_L are invariants: if $\Theta_L(\tau) \neq \Theta_{L'}(\tau)$ then L, L' cannot be isometric. What about the converse? The converse fails in general.

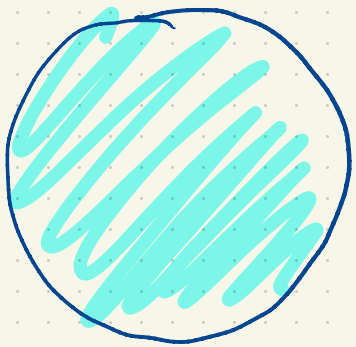
First known example ($n=16$): There are two lattices $E_8 \oplus E_8$ and D_{16} in $\mathbb{R}^{16}$ which are not isometric but they have the same theta-series

$$1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^{2n} \qquad \sigma_7(n) = \sum d^7 \quad \text{(sum over positive integers } d \mid n\text{)}.$$

This appears in J.P. Serre, A Course in Arithmetic and in a very short paper J. Milnor. (1964)

"Can you hear the shape of a drum?" (question by Mark Kac) $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$





Spectrum of a domain $\Omega \subset \mathbb{R}^2$ is the set of eigenvalues of Δ

$$\Delta f = \lambda f$$
$$f|_{\partial\Omega} = 0 \quad \text{(Dirichlet boundary conditions)}$$

$\mathbb{R}^{16}/L$ $\qquad$ $L \subset \mathbb{R}^{16}$ lattice

The eigenvalues of the Laplacian for the n-dimensional torus $\mathbb{R}^n/L$ can be immediately read off from the theta series of the lattice L.

Given a prime $p \equiv 1 \bmod 4$, the number of vectors $(x_1, \dots, x_6) \in \mathbb{Z}^6$ satisfying

$$x_1^2 + \cdots + x_6^2 = 6p, \qquad x_i \equiv 1 \bmod 4$$

is $p^2 + 1$.

$p = 5$: $x_i \equiv 1 \bmod 4$ $(x_i \in \{1, -3, 5, -7, 9, -11, \dots\})$

$x_1^2 + \cdots + x_6^2 = 30$

$(5, 1^5) = (5, 1, 1, 1, 1, 1)$	(6 such)
$(-5^3, 1^3)$	$\left(\binom{6}{3} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20 \text{ such}\right)$
	26 solutions

$p = 13$: $x_i \equiv 1 \bmod 4$ $(x_i \equiv 1, -3, 5, -7)$

$x_1^2 + \cdots + x_6^2 = 78$

$(-7, 5, 1^4)$	30 such
$(-7, -3^3, 1^2)$	60 such
$(5^3, 1^3)$	20 such
$(5^2, -3^3, 1)$	60 such
	170 solutions

Geometry: The set of solutions has the property that for any two distinct solutions $x = (x_1, \dots, x_6)$, $y = (y_1, \dots, y_6)$,

$x \cdot y \not\equiv 0 \bmod p$.

Another similar construction: Given $p \equiv 1 \bmod 4$ prime, list all vectors $x = (x_1, \cdots, x_6) \in \mathbb{Z}^6$ satisfying

- $x_1^2 + \cdots + x_6^2 = p$
- $x_1 + 1 \equiv x_2 \equiv x_3 \equiv x_4 \equiv x_5 \equiv x_6 \bmod 2$

We only care about pairs of vectors $\pm x$

$p=5$: $\pm(0|1,\pm1^4)$ (16 such) i.e. 16 pairs $\pm x$

$\pm(1|\pm2,0^4)$ (10 such) i.e. 10 pairs $\pm x$

26 pairs

$p=13$: $\pm(0|3,\pm1^4)$ $5\times16 = 80$ such

$\pm(1|\pm2^3,0^2)$ $\binom{5}{2}\times 2^3 = 80$ such

$\pm(3|\pm2,0^4)$ $5\times2 = 10$ such

170 such

In general there are p^2+1 solutions.

The proof in each case relies on modular forms. We can prove the formula using $\Theta_{E_8}(\tau) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^{2n}$. The connection uses geometry in E_8/pE_8, an 8-dimensional vector space over $\mathbb{F}_p$. An ovoid is a set of p^3+1 points x_i $(i = 1, 2, \cdots, p^3+1)$ such that

$x_i \cdot x_i = 0$; $x_i \cdot x_j \neq 0$ for $i \neq j$. $\sigma_3(p) = 1^3 + p^3 = p^3+1$.

Why do we care about ovoids? Every ovoid in $\mathbb{F}_p^8$ gives rise to "many" ovoids in $\mathbb{F}_p^6$ by "slicing". An ovoid in $\mathbb{F}_p^6$ has p^2+1 points. These give projective planes of order p^2.

p^4+p^2+1 points / lines

Every pt on p^2+1 lines

" line has p^2+1 pts

"Most" of the known finite planes arise from this construction. This is a heuristic argument only.

↑ explicitly constructed.

Original construction of ovoids in $\mathbb{F}_p^8$ from E_8 is due to Conway, Kleidman, Wilson (1988?)

Conway conjectured that $\lim_{n\to\infty} \frac{\text{no. of known planes of order} \le n \text{ arising from } E_8\text{-type ovoids}}{\text{no. of known planes of order} \le n} = 1$

M. (1991) generalized Conway's construction giving many more ovoids from E_8.

Used $\Theta_{E_8}(\tau) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^{2n}$ and $\Theta_{E_8\oplus E_8}(\tau) = 1 + 480\sum_{n=1}^{\infty}\sigma_7(n)q^{2n}$

$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots$

$j\left(\frac{1+\sqrt{-163}}{2}\right) = -640320^3 = -262537412640768000$ $\qquad q = e^{\pi i \tau}$

gives rise $e^{\pi\sqrt{163}} \approx 640320^3 + 743 + 0.999999999999250\ldots$

$\mathbb{Q}[\sqrt{-163}]$ has unique factorization

Monday: applications of modular forms to elliptic curves