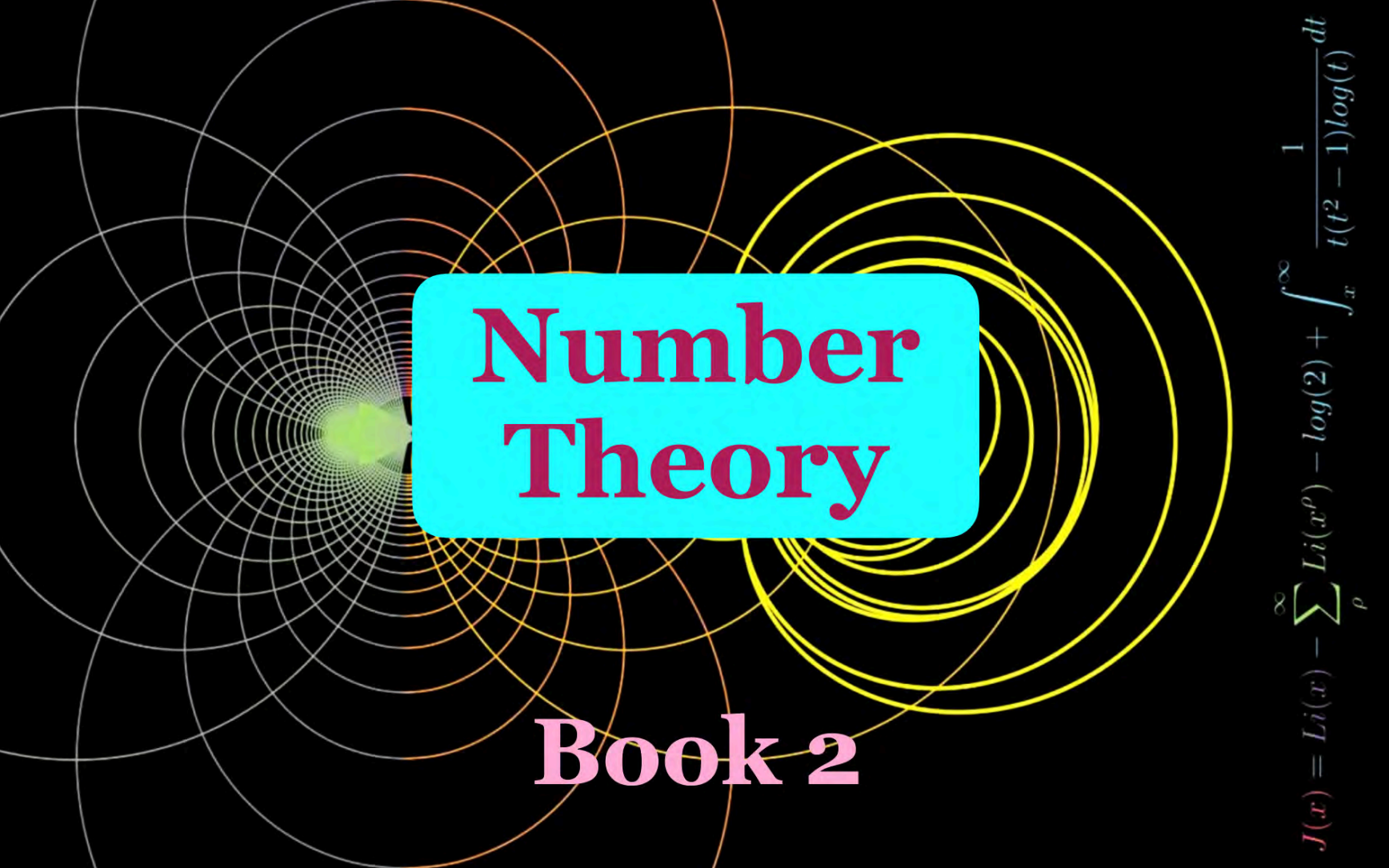

Number
Theory
Book 2
$J(x) = Li(x) - \sum_{\rho}^{\infty} Li(x^{\rho}) - \log(2) + \int_{x}^{\infty} \frac{1}{t(t^2-1)\log(t)} dt$

If R is any ring with identity then $R^\times = \{\text{units in } R\} = \{\text{invertible elements in } R\} = \{u \in R : uv = vu = 1 \text{ for some } v \in R\}$.
The units in the ring of $n \times n$ matrices over R form a mult. gp. $GL_n(R)$.
The units $R^\times$ the unit group of R.

If R is a commutative ring with identity then $R^\times$ is abelian.

Take $\mathcal{O} = \{\text{alg. integers in } K\}$ $K \supseteq \mathbb{Q}$ finite extension. We want to describe $\mathcal{O}^\times$ = unit group of the extension, an abelian multiplicative group.

eg. $\mathbb{Z}^\times = \{\pm 1\}$. $|\mathcal{O}^\times| \geq 2$ since $\pm 1 \in \mathcal{O}^\times$.

In a real quadratic field, $\mathcal{O}^\times = \{\pm 1\} \times \langle \alpha \rangle$ (can)

$\{\pm 1\}$: torsion part: the elements of finite order in $\mathcal{O}$ (roots of unity in $\mathcal{O}$)

$\langle \alpha \rangle = \{\alpha^k : k \in \mathbb{Z}\}$: infinite cyclic group with generator α

eg. $K = \mathbb{Q}[\sqrt{2}]$, $\mathcal{O} = \mathbb{Z}[\sqrt{2}]$,
$\mathcal{O}^\times = \{\pm(1+\sqrt{2})^k : k \in \mathbb{Z}\}$

$\alpha = 1+\sqrt{2}$ is a generator of the "infinite" part of $\mathcal{O}^\times$. ← fundamental unit.

Note: $\alpha^{-1} = -1+\sqrt{2}$

$\mathcal{O}^\times = \{\pm 1, \pm 1 \pm \sqrt{2}, \pm 3 \pm 2\sqrt{2}, \cdots\}$

$\langle \alpha \rangle = \langle 1+\sqrt{2} \rangle$ = positive elements in $\mathcal{O}^\times$.

Solutions of $x^2 - 2y^2 = \pm 1$ are $\{(\pm 1, 0), (\pm 1, \pm 1), (\pm 3, \pm 2), \cdots\}$.

$\{\pm 1\} \times \langle 1-\sqrt{2} \rangle = \{\pm 1\} \times \langle 1+\sqrt{2} \rangle = \mathcal{O}^\times$

$\langle 1-\sqrt{2} \rangle$: not canonical; $\langle 1+\sqrt{2} \rangle$: canonical

$\mathcal{O}^\times = \{\text{units}\} = \{\text{solutions of Pell's equation } x^2 - dy^2 = \pm 1\}$

Imaginary quadratic fields $K = \mathbb{Q}[\sqrt{d}]$, $d < 0$

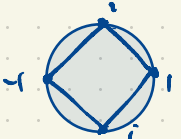
$\mathcal{O}^\times$ is finite since the equation $x^2 - dy^2 = \pm 1$ has only finitely many solutions

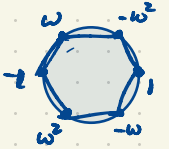
If $K = \mathbb{Q}[\sqrt{-5}]$ then $\mathcal{O} = \mathbb{Z}[\sqrt{-5}]$, $\mathcal{O}^\times = \{\pm 1\}$.

$\mathbb{Q}[\sqrt{-3}]$ $\mathcal{O} = \mathbb{Z}[\omega]$, $\omega = \frac{-1+\sqrt{-3}}{2}$ $\mathcal{O}^\times = \{\pm 1, \pm\omega, \pm\omega^2\}$ sixth roots of unity

$\mathbb{Q}[\sqrt{-1}] = \mathbb{Q}[i]$ $\mathcal{O} = \mathbb{Z}[i]$ $\mathcal{O}^\times = \{\pm 1, \pm i\}$

$\omega^2 = \bar{\omega} = \frac{-1-\sqrt{-3}}{2}$





Dirichlet's Unit Theorem If $\mathcal{O}$ is the ring of integers in a number field $K \supseteq \mathbb{Q}$ ($[K:\mathbb{Q}] < \infty$), then

$$\mathcal{O}^\times = \{\text{roots of unity in } \mathcal{O}\} \times \text{free abelian group of rank } r_1+r_2-1$$

($\{\text{roots of unity in } \mathcal{O}\}$ = torsion part of $\mathcal{O}^\times$; finite cyclic group of even order)

$$\cong \mathbb{Z}^{r_1+r_2-1} = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{r_1+r_2-1}$$

written additively.

r_1+r_2-1 = number of generators.

Every number field K can be embedded $K \hookrightarrow \mathbb{C}$ (one-to-one homomorphism of rings) in $n = [K:\mathbb{Q}]$ distinct ways. r_1 of these embeddings have their image $\subset \mathbb{R}$; the other $2r_2$ such embeddings non-real. K has r_1 real and $2r_2$ non-real embeddings.

What are r_1, r_2?

$K \supseteq \mathbb{Q}$ is a number field. $K = \mathbb{Q}[\alpha]$ for some $\alpha \in K$. (Not canonical.)

$$\cong \mathbb{Q}[x]/(m(x))$$

$m(x) \in \mathbb{Q}[x]$ irreducible

has r_1 real roots, $2r_2$ non-real roots (r_2 complex conjugate pairs)

$n = r_1 + 2r_2$.

eg. $\mathbb{Q}[\sqrt{2},\sqrt{3}] = \{a+b\sqrt{2}+c\sqrt{3}+d\sqrt{6} : a,b,c,d \in \mathbb{Q}\}$ degree 4 over $\mathbb{Q}$

$= \mathbb{Q}[\sqrt{2}+\sqrt{3}]$

K can be embedded in $\mathbb{C}$ in $n = [K:\mathbb{Q}]$ ways (not canonically), $n = r_1 + 2r_2$.

by mapping $x \mapsto$ any of the roots of $m(x)$.

Dirichlet's Unit Theorem applies to all number fields, Galois or not. $|\mathrm{Aut}\, K| \le n$. Equality iff K is a Galois extension.

Eg. $K = \mathbb{Q}[\sqrt{d}]$, d squarefree, $d \neq 0,1$. $K \cong \mathbb{Q}[x]/(x^2-d)$

If $d>0$ then $r_1=2$, $r_2=0$, $n = r_1+2r_2 = 2$, rank $r_1+r_2-1 = 2+0-1 = 1$.

If $d<0$ then $r_1=0$, $r_2=1$, $n = r_1+2r_2 = 2$, rank $r_1+r_2-1 = 0+1-1 = 0$.

Another example with Dirichlet's Unit Theorem

$K = \mathbb{Q}[\sqrt{2},\sqrt{3}] = \{a+b\sqrt{2}+c\sqrt{3}+d\sqrt{6}\} \supset \mathbb{Q}$ degree $[K:\mathbb{Q}] = 4$

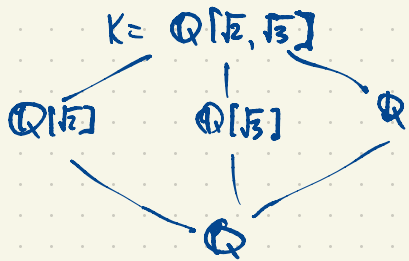
Every embedding $K \hookrightarrow \mathbb{C}$ is real ($K \hookrightarrow \mathbb{R}$) i.e. $r_1 = 4$, $r_2 = 0$, $n = r_1 + 2r_2 = 4+0 = 4$.

Dirichlet's unit theorem: $\mathcal{O}^\times \cong \{\pm 1\} \times \mathbb{Z}^3$ ← the only roots of $\mathbb{R}$ are ± 1

i.e. $\mathcal{O}^\times = \{\pm \alpha^i \beta^j \gamma^k : i,j,k \in \mathbb{Z}\}$

$r_1 + r_2 - 1 = 1+0-1 = 3$ gives the rank.

What are the generators α, β, γ unity in this case? (fundamental units)



These are all the five subfields of K by Galois theory.

$\pm(1+\sqrt{2})^k$ units in $\mathbb{Q}[\sqrt{2}]$

$\pm(2+\sqrt{3})^k$ ---- $\mathbb{Q}[\sqrt{3}]$

$\pm(5+2\sqrt{6})^k$ --- $\mathbb{Q}[\sqrt{6}]$

$x^2 - 2y^2 = \pm 1 = (1+\sqrt{2})(-1+\sqrt{2}) = -1$

$(1+\sqrt{2})^k(-1+\sqrt{2})^k = 1$

$(2+\sqrt{3})(2-\sqrt{3}) = 1$

$x^2 - 6y^2 = \pm 1$ (5,2) fundamental solution

$(5+2\sqrt{6})(5-2\sqrt{6}) = 1$

Is $\mathcal{O}^\times = \pm(1+\sqrt{2})^k(2+\sqrt{3})^l(5+2\sqrt{6})^m$, $k,l,m \in \mathbb{Z}$?

No, these are only 25% of the units in K.

First of all, $\mathcal{O} = \{\text{alg. int. in } K\} \supseteq \{a+b\sqrt{2}+c\sqrt{3}+d\sqrt{6} : a,b,c,d \in \mathbb{Z}\}$. Since $1, \sqrt{2}, \sqrt{3}, \sqrt{6}$ are alg. int.

Actually $\mathcal{O} = (1, \sqrt{2}, \sqrt{3}, \sqrt{6}, \frac{\sqrt{2}+\sqrt{6}}{2}) = (1, \sqrt{2}, \sqrt{3}, \frac{\sqrt{2}+\sqrt{6}}{2})$

If $\alpha = \frac{\sqrt{2}+\sqrt{6}}{2}$ then $\alpha^2 = \frac{2+6+4\sqrt{3}}{4} = 2+\sqrt{3}$, $\alpha^2 - 2 = \sqrt{3}$, $\alpha^4 - 4\alpha^2 + 1 = 0$. $\Rightarrow \alpha$ is an alg. int.

In fact $\alpha \in \mathcal{O}^\times$.

$\beta = \sqrt{2}+\sqrt{3}$ then $\beta^2 = (\sqrt{2}+\sqrt{3})^2 = 2+3+2\sqrt{6} = 5+2\sqrt{6}$. $\beta^2 - 5 = 2\sqrt{6} \Rightarrow \beta^4 - 10\beta^2 + 1 = 0$.

$\gamma = 1+\sqrt{2}$ $(1+\sqrt{2})(-1+\sqrt{2}) = 1$. $1 = 10\beta^2 - \beta^4 = \beta \cdot (10\beta - \beta^3)$

Checked using PARI/GP.

$f: \mathbb{C} \to \mathbb{C}$ is analytic in a region $\Omega \subset \mathbb{C}$ (open set) if f' exists in Ω.
In this case at every point $z_0 \in \Omega$ there is a series expansion $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ in some disk $|z-z_0| < r$ in Ω.

A function f is meromorphic in Ω if at every point $z_0 \in \Omega$ it has a Laurent expansion $f(z) = \sum_{n=-k}^{\infty} a_n (z-z_0)^n$, $k \in \mathbb{Z}$
When $a_{-k} \neq 0$ with $k > 0$, we have a pole of order k (assuming k is the largest such)

$e^{\frac{1}{z}}$ has an essential singularity at 0 ("worse" than a pole)

f has a simple pole at z_0 if $f(z) = \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$ $0 < |z-z_0| < r$ in Ω

a_{-1} = Residue of f at z_0.

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = a_{-1} = \lim_{z \to z_0} (z-z_0) f(z)$$

$\zeta(s)$, $\zeta_K(s)$ is meromorphic in $\mathbb{C}$ with a simple pole at $s = 1$.

Class number formula
$$\lim_{s \to 1} (s-1)\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2} \operatorname{Reg}_K h_K}{w_K \sqrt{|\operatorname{disc} K|}}$$
(residue of $\zeta_K(s)$ at its simple pole)

Reg_K = regulator of K
h_K = class number
w_K = number of roots of unity in K.
r_1 = no. of real embeddings $K \hookrightarrow \mathbb{R}$
$2r_2$ = " " non-real embeddings $K \hookrightarrow \mathbb{C}$
$n = [K:\mathbb{Q}] = r_1 + 2r_2$

Every number field $K \supseteq \mathbb{Q}$ has the form $K = \mathbb{Q}[\alpha] \cong \mathbb{Q}[x]/(m(x))$ $m(x)$ = min. poly. of α over $\mathbb{Q}$.

eg. $K = \mathbb{Q}[\sqrt{2}, \sqrt{3}]$ has $r_1 = 4$ real embeddings $K \hookrightarrow \mathbb{R}$
$r_2 = 0$ non-real ... $K \hookrightarrow \mathbb{C}$

eg. $\alpha = \sqrt{2} + \sqrt{3}$ is a generator
$K = \mathbb{Q}[\alpha] = \mathbb{Q}[x]/(x^4 - 10x^2 + 1)$
$r_1 = 4$ real roots $\pm\sqrt{2} \pm \sqrt{3}$
$2r_2 = 0$ non-real roots.

I worked this out with $K = \mathbb{Q}[\sqrt{-5}]$ $\mathrm{Reg}_K = 1$ in this case $h_K = 2$.

Remarks about computation:

$\zeta_K(2) \approx 1.855557$

$1/\zeta_K(2) \approx 0.53892$

$\frac{\zeta_K(s)}{\zeta(s)}$ has no pole at 1. It's a Dirichlet L-function. $= (1 - \frac{1}{3^s})^{-1}(1 - \frac{1}{7^s})^{-1}(1 + \frac{1}{11^s})^{-1}(1 + \frac{1}{13^s})^{-1}(1 + \frac{1}{17^s})^{-1}(1 + \frac{1}{19^s})^{-1}(1 - \frac{1}{23^s})^{-1} \cdots$

The Riemann zeta function $\zeta(s) = \frac{1}{s-1} + O(1)$ as $s \to 1$ i.e. Residue of $\zeta(s)$ at $s = 1$ is 1.

$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$ converges by comparison with $\int_1^{\infty} \frac{1}{t^x} dt = \frac{1}{x-1}$. $\frac{1}{x-1} < \zeta(x) < \frac{1}{x-1} + 1$.

($x > 1$ real)

Remarks about class number h_K of a quadratic number field $K = \mathbb{Q}[\sqrt{d}]$:

We know only finitely many imaginary quadratic fields have class number 1, for

$d = -1, -2, -3, -7, -11, -19, -43, -67, -163$ eg. $h_{\mathbb{Q}[\sqrt{-5}]} = 2$

In fact $h_K \to \infty$ as $d \to -\infty$

We know much less about the real quadratic fields.

We think there are infinitely many real quadratic fields with class number 1.

$e^{\pi\sqrt{163}} = 262\,537\,412\,640\,768\,743.999\,999\,999\,999\,250\,725\,9\ldots$

- $x^2 + x + 41$ has prime values for $x = 0, 1, 2, 3, \cdots, 39$
- There is no nonconstant polynomial $f(x) \in \mathbb{Z}[x]$ having only prime values.
- There is no known polynomial $f(x) \in \mathbb{Z}[x]$ of degree > 1 which is prime infinitely often.
- Is x^2+1 prime infinitely often? Open problem.
- $\operatorname{disc}(x^2 + x + 41) = 1 - 4 \cdot 41 = -163$.

$640320^3 + 744 = 262\,537\,412\,640\,768\,744$

The polynomial $x^2 + x + k$ has prime values for $x = 0, 1, 2, \cdots, k-2$ $(k > 0)$

$\iff k \in \{1, 2, 3, 5, 11, 17, 41\}$

$\iff h_{\mathbb{Q}[\sqrt{d}]} = 1, \quad d = 1 - 4k$

$\iff d \in \{-3, -7, -11, -19, -43, -67, -163\}$

Where do Dirichlet L-functions come from?

Dirichlet characters are functions $\chi: \mathbb{Z} \to \mathbb{C}$ satisfying $\chi(ab) = \chi(a)\chi(b)$ and some additional properties $\chi(1) = 1$, $\chi(0) = 0$ unless $\chi = 1$ identically (we usually ignore this case)

$\chi(1) = \chi(1)\chi(1) \Rightarrow \chi(1) = 0$ or 1.

χ should be a function on $\mathbb{Z}/n\mathbb{Z}$ ie. $\chi(a) = \chi(b)$ whenever $a \equiv b \bmod n$.

$$\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \xrightarrow{\chi} \mathbb{C}$$

(χ is a Dirichlet character mod n in this case)

The Dirichlet L-function corresponding to χ is

$$L_\chi(s) = L(\chi, s) = \sum_{k=1}^{\infty} \frac{\chi(k)}{k^s} = \prod_p \left(\frac{1}{1 - \frac{\chi(p)}{p^s}}\right)$$

$$= \frac{\xi_K(s)}{\zeta(s)}$$

Euler factorization

proof: use FTA (Fund. Thm. of Arithmetic: unique factorization in $\mathbb{Z}$)

$$\frac{1}{1 - \frac{a}{p^s}} = 1 + \frac{a}{p^s} + \frac{a^2}{p^{2s}} + \frac{a^3}{p^{3s}} + \cdots$$

eg. There are $\phi(20) = 8$ Dirichlet characters mod 20 including

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25 ⋯
$\chi(k)$	0	1	0	1	0	0	0	1	0	1	0	−1	0	−1	0	0	0	−1	0	−1	0	1	0	1	0	0

$$L(\chi, s) = 1 + \frac{1}{3^s} + \frac{1}{7^s} + \frac{1}{9^s} - \frac{1}{11^s} - \frac{1}{13^s} - \frac{1}{17^s} + \frac{1}{21^s} + \frac{1}{23^s} + \frac{1}{27^s} + \frac{1}{29^s} - \frac{1}{31^s} - \frac{1}{33^s} - \cdots$$

$$\zeta(s) = \prod_p \frac{1}{1 - \frac{1}{p^s}}$$

$$\xi_K(s) = \prod_{\mathfrak{p}} \frac{1}{1 - \frac{1}{N(\mathfrak{p})^s}}$$

For $p = 2$, $(2) = \mathfrak{p}_2^2$, $\mathfrak{p}_2 = (2, 1+\sqrt{-5})$, $N(\mathfrak{p}_2) = 2$,

Same at $p = 5$.

For $p = 3$, $(3) = \mathfrak{p}_3\bar{\mathfrak{p}}_3$, $\mathfrak{p}_3 = (3, 1+\sqrt{-5})$, $N(\mathfrak{p}_3) = 3$

Same for $p \equiv 1, 3, 7, 9 \bmod 20$ (p splits)

For $p = 11$, $(11) = \mathfrak{p}_{11}$, $N(\mathfrak{p}_{11}) = 11^2 = 121$

Same for $p \equiv 11, 13, 17, 19 \bmod 20$ (p remains prime)

$\frac{\xi_K(s)}{\zeta(s)}$ has Euler factors at 2: $\frac{1/(1-\frac{1}{2^s})}{1/(1-\frac{1}{2^s})} = 1 = \frac{1}{1-\frac{0}{2^s}}$

— — — — 3: $\frac{1/(1-\frac{1}{3^s})^2}{1/(1-\frac{1}{3^s})} = \frac{1}{1-\frac{1}{3^s}}$

$\frac{\xi_K(s)}{\zeta(s)}$ ⋯⋯ 11: $\frac{1/(1-\frac{1}{121^s})}{1/(1-\frac{1}{11^s})} = \frac{1}{1+\frac{1}{11^s}}$

Why do we care about Dirichlet L-functions?

These are essential for proving:

Dirichlet's Theorem: Every arithmetic progression $a, a+k, a+2k, a+3k, a+4k, \ldots$ ($a, k \in \mathbb{Z}$, $k>0$, $\gcd(a,k)=1$) contains infinitely many primes i.e. there are infinitely many primes $\equiv k \bmod a$.

The number of Dirichlet characters mod 5 is $4 = \phi(5)$

Let χ be a Dirichlet character mod 5: $\chi(a)$ only depends on $a \bmod 5$.

$\chi(ab) = \chi(a)\chi(b)$

$\chi(1) = 1$.

$\chi(2)^4 = \chi(2^4) = \chi(16) = \chi(1) = 1 \Rightarrow \chi(2) \in \{\pm 1, \pm i\}$

$\chi(4) = \chi(2)^2$

$\chi(3) = \chi(2)^3$

$\chi(1) = 1$

n	0	1	2	3	4	5	6	7	...
$\chi_0(n)$	0	1	1	1	1	0	1	1	...
$\chi_1(n)$	0	1	i	$-i$	-1	0	1	i	...
$\chi_2(n)$	0	1	-1	-1	1	0	1	-1	...
$\chi_3(n)$	0	1	$-i$	i	-1	0	1	$-i$	...

$\chi(k)$ is either 0 (if $\gcd(k,n)>1$) or a root of unity (if $\gcd(k,n)=1$)

We'll discuss the special case of Dirichlet's Theorem for $n=4$:

There are infinitely many primes $\equiv 1 \bmod 4$ and

- - - - - - - $\equiv 3 \bmod 4$.

Warm-up: There are infinitely many primes.

- Euclid's proof
- Euler's proof: $\sum_p \frac{1}{p}$ diverges. (Sum over primes p)

$= \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \cdots$