

Number Theory

Book 1

$$J(x) = Li(x) - \sum_{\rho} Li(x^{\rho}) - \log(2) + \int_x^{\infty} \frac{1}{t(t^2 - 1)\log(t)} dt$$

Rational integers ("ordinary integers") $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$

(Algebraic) integers: A number $\alpha \in \mathbb{C}$ is algebraic if it is a root of a nonzero poly. with coefficients in \mathbb{Q}

i.e. $f(\alpha) = 0$ for some $f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$, $a_i \in \mathbb{Q}$ i.e. $f(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$, $a_i \in \mathbb{Z}$ not all zero.

eg. $\frac{\sqrt{2}}{7}$ is algebraic since it's a root of $x^2 - \frac{2}{49} \in \mathbb{Q}[x]$ or $49x^2 - 2 \in \mathbb{Z}[x]$.

We say α is an (algebraic) integer if α is a root of a monic poly. with coefficients in \mathbb{Z} i.e.

$f(\alpha) = 0$ for some $f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$, $a_i \in \mathbb{Z}$

eg. $\sqrt{2}$ is integral (it's an ^(algebraic) integer)

If $\mathcal{O} = \{\alpha \in \mathbb{C} : \alpha \text{ is an algebraic integer}\} \subset \mathcal{A} \subset \mathbb{C}$

$\mathcal{A} = \{\alpha \in \mathbb{C} : \alpha \text{ is algebraic}\} \subset \mathbb{C}$

$\mathcal{O} \cap \mathbb{Q} = \mathbb{Z}$ (the rational integers are the ordinary integers).

Fermat's equation: $x^2 - 5y^2 = 1$ has infinitely many ^(ordinary) integer solutions.

$x^2 - 5y^2 = 1 \iff (x + y\sqrt{5})(x - y\sqrt{5}) = 1$ where $x + y\sqrt{5} \in \mathcal{O}$ has Norm is $N(x + y\sqrt{5}) = (x + y\sqrt{5})(x - y\sqrt{5}) = x^2 - 5y^2$

$N(\alpha\beta) = N(\alpha)N(\beta)$ so if $\alpha = 9 + 4\sqrt{5}$ then $N(\alpha) = 9^2 - 5 \cdot 4^2 = 1$ so $N(\alpha^k) = N(\alpha)^k = 1^k = 1$

eg. $(9 + 4\sqrt{5})^2 = 81 + 80 + 72\sqrt{5} = 161 + 72\sqrt{5}$ also has norm 1 so $161^2 - 5 \cdot 72^2 = 1$

$(9 + 4\sqrt{5})^3 = (161 + 72\sqrt{5})(9 + 4\sqrt{5}) = 2889 + 1292\sqrt{5}$

\mathcal{A} is the fraction field of \mathcal{O} i.e. $\mathcal{A} = \{\frac{\alpha}{\beta} : \alpha, \beta \in \mathcal{O}, \beta \neq 0\}$

$\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}$.

eg. $\sqrt{2}, \sqrt{3} \in \mathcal{O}$ (root of $x^2 - 2$, $x^2 - 3$ respectively)

$\Rightarrow \sqrt{2} \cdot \sqrt{3} = \sqrt{6} \in \mathcal{O}$ (root of $x^2 - 6$)

$\sqrt{2} + \sqrt{3} \in \mathcal{O}$ (root of $x^2 - 10x^2 + 1$)

the min. poly. of $\sqrt{2} + \sqrt{3}$ over \mathbb{Q} .

\mathcal{O} is the ring of alg. int.

\mathcal{A} is the field of alg. numbers.

x	y
-1	8
± 9	± 4
± 161	± 72
± 2889	± 1292
\vdots	\vdots

$$x = \sqrt{2} + \sqrt{3}$$

$$x^2 = 5 + 2\sqrt{6}$$

$$x^2 - 5 = 2\sqrt{6}$$

$$x^4 - 10x^2 + 25 = 24$$

$$x^4 - 10x^2 + 1 = 0$$

Let $\alpha \in \mathbb{C}$. If α is algebraic then it has a minimal poly. over \mathbb{Q} i.e. $m(x) \in \mathbb{Q}[x]$ is monic with $m(\alpha) = 0$ and $\deg m(x)$ is as small as possible. In this case $m(x)$ is unique. All polynomials in $\mathbb{Q}[x]$ having α as a root are multiples of $m(x)$.

$$(m(x)) = \{h(x)m(x) : h(x) \in \mathbb{Q}[x]\}$$

$\mathbb{Q}[x]$ is a principal ideal ring (every ideal is principal).

Review: Let R be a commutative ring with identity $1 \in R$. (eg. \mathbb{Z} , $\mathbb{Q}[x]$).

An ideal is a subset $J \subseteq R$, $0 \in J$ such that J is closed under R -linear combinations i.e. $ra + sb \in J$ for all $r, s \in R$, $a, b \in J$. (Every ideal is a subring but not conversely).

Given $a_1, \dots, a_k \in R$, these elements generate an ideal $(a_1, a_2, \dots, a_k) = \{r_1 a_1 + r_2 a_2 + \dots + r_k a_k : r_1, \dots, r_k \in R\} \subset R$.

eg. in \mathbb{Z} , $(2, 6) = \{2r + 6s : r, s \in \mathbb{Z}\} = (3)$

Every ideal in \mathbb{Z} is principal i.e. generated by a single element.

Return to previous setting $\alpha \in \mathbb{C}$ algebraic.

There is a ring homomorphism $\mathbb{Q}[x] \rightarrow \mathbb{C}$

$$f(x) \mapsto f(\alpha)$$

with kernel $J = \{f(x) \in \mathbb{Q}[x] : f(\alpha) = 0\} \subset \mathbb{Q}[x]$ an ideal.

Since α is algebraic, $J = (m(x))$ with $m(x) \neq 0$. Scale $m(x)$ if needed to get $m(x)$ monic. Then $m(x)$ is unique; it's the minimal poly. of α over \mathbb{Q} .

$\mathbb{Z}[x]$ is not a principal ideal ring

$\mathbb{Q}[x, y]$

In $\mathbb{Q}[x, y]$, $(x^2, xy, y^2) = \{f(x, y) \in \mathbb{Q}[x, y] \text{ with no const. term, no } x \text{ term, no } y \text{ term}\}$ is a non-principal ideal.

This ideal cannot be generated by 1 or 2 generators; you need at least 3 generators to generate it.

In $\mathbb{Q}[x, y]$, every ideal is finitely generated (there is a finite list of generators) (Hilbert's basis theorem)

In number theory, a number field ^(algebraic) is a finite extension $K \supseteq \mathbb{Q}$ eg. $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\} \supset \mathbb{Q}$ is a quadratic extension.

$[K:\mathbb{Q}] = \text{degree of the extension}$

$[E:F] = \text{degree of field extension } E \supseteq F \text{ (} F \text{ subfield of } E\text{)}$
 $= \text{dimension of } E \text{ as a vector space over } F$

$[\mathbb{Q}[\sqrt{2}]:\mathbb{Q}] = 2$ since $\{1, \sqrt{2}\}$ is a basis for $\mathbb{Q}[\sqrt{2}]$ over \mathbb{Q} .

$\mathbb{C} \supset \mathbb{R} \supset \mathbb{Q}$

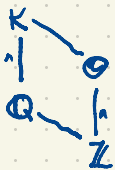
$[\mathbb{C}:\mathbb{R}] = 2$

$[\mathbb{C}:\mathbb{Q}] = \infty$

$[\mathbb{R}:\mathbb{Q}] = \infty$.

$K \supseteq \mathbb{Q}$ finite extension $n = [K:\mathbb{Q}] < \infty$. All elements $\alpha \in K$ are algebraic i.e. $K \subset \mathcal{A} = \{\text{algebraic numbers}\}$.

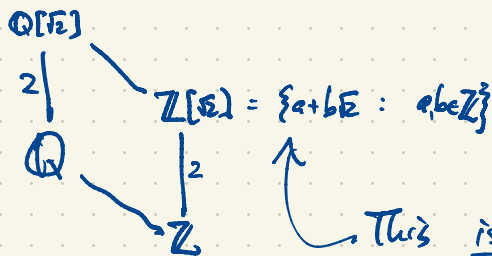
If $\alpha \in K$ then $1, \alpha, \alpha^2, \dots, \alpha^n$ so there exist $a_0, a_1, \dots, a_n \in \mathbb{Q}$ not all zero, s.t. $a_0 + a_1\alpha + \dots + a_n\alpha^n = 0$
so α is algebraic. (of degree $\leq n$). The degree of an algebraic number is the degree of its min. poly. $= [\mathbb{Q}(\alpha):\mathbb{Q}]$



$\mathcal{O} = \{\text{alg. integers in } K\}$

$K = \left\{ \frac{\alpha}{\beta} : \alpha, \beta \in \mathcal{O}, \beta \neq 0 \right\}$.

eg.



This is a principal ideal ring.

$\mathcal{O} = \{r_1\alpha_1 + \dots + r_n\alpha_n : r_i \in \mathbb{Z}\}$

$\alpha_1, \dots, \alpha_n$ base for $\mathcal{O} \supseteq \mathbb{Z}$

$\alpha_1, \dots, \alpha_n$ basis for $K \supseteq \mathbb{Q}$

\mathcal{O} is not always (usually) a principal ideal ring.

Every ideal $J \subset \mathcal{O}$ has the form

$J = (a)$ or (a, b)

$\mathbb{Z}[\sqrt{2}]$ has infinitely many units

$$(3+2\sqrt{2})(3-2\sqrt{2}) = 1$$

units in $\mathbb{Z}[\sqrt{2}]$

$$\mathbb{Z}[\sqrt{2}]^\times = \{\pm (3+2\sqrt{2})^n : n \in \mathbb{Z}\} \cong (\mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}$$

Ex. $\mathbb{Q}[\sqrt{5}] \supset \mathbb{Q}$ is another quadratic extension (quadratic number field i.e. $[\mathbb{Q}[\sqrt{5}]:\mathbb{Q}] = 2$)
 $\{1, \sqrt{5}\}$ basis for the extension

$\mathbb{Q}[\sqrt{5}]$

$$\mathbb{Z} \mid \mathbb{Q}[\sqrt{5}] = \{a + b\sqrt{5} : a, b \in \mathbb{Z}\}$$



\mathbb{O}_K (commutative ring with identity)

$\alpha \in \mathbb{O}^* = \{\text{units of } \mathbb{O}\}$ iff $\alpha\beta = 1$ for some $\beta \in \mathbb{O}$.

\mathbb{O}^* is a multiplicative group (abelian).

The only units in $\mathbb{Z}[\sqrt{5}]$ are $\mathbb{Z}[\sqrt{5}]^* = \{\pm 1\}$.

The norm of $\alpha \in \mathbb{Z}[\sqrt{5}]$ is $N(\alpha) = \alpha\bar{\alpha}$, $a + b\sqrt{5} = a - b\sqrt{5}$

$$= (a + b\sqrt{5})(a - b\sqrt{5})$$

For $\alpha, \beta \in \mathbb{Z}[\sqrt{5}]$, $N(\alpha\beta) = N(\alpha)N(\beta)$.

$$(\alpha\beta)(\overline{\alpha\beta}) = (\alpha\bar{\alpha})(\beta\bar{\beta})$$

If $\alpha = a + b\sqrt{5} \in \mathbb{Z}[\sqrt{5}]^*$ then $N(\alpha) = a^2 + 5b^2 \in \{0, 1, 2, 3, \dots\}$ since $a, b \in \mathbb{Z}$.

$\alpha\beta = 1$ for some $\beta \in \mathbb{Z}[\sqrt{5}]$

$$N(\alpha)N(\beta) = N(\alpha\beta) = N(1) = 1 \Rightarrow N(\alpha) = N(\beta) = 1$$

$\alpha \in \mathbb{Z}[\sqrt{5}]$

α is reducible if $\alpha = \alpha_1\alpha_2$, $\alpha_1, \alpha_2 \in \mathbb{Z}[\sqrt{5}]$, neither α_1 nor α_2 is a unit.

α is irreducible if $\left\{ \begin{array}{l} \text{the only way to factor } \alpha = \alpha_1\alpha_2, \alpha_1, \alpha_2 \in \mathbb{Z}[\sqrt{5}] \\ \text{and } \alpha \text{ is not a unit} \end{array} \right.$ is if one of α_1, α_2 is a unit.

\mathbb{Z} has unique factorization

$$\begin{aligned} 12 &= 2 \times 2 \times 3 \\ &= (-2) \times 2 \times (-3) \\ &= 2 \times 3 \times 2 \\ &= (-2) \times 3 \times (-2) \end{aligned}$$

\mathbb{Z} has irreducible elements

$$\pm 2, \pm 3, \pm 5, \pm 7, \pm 11, \dots$$

\mathbb{Z} has units ± 1 (invertible elements)

In $\mathbb{Z}[\sqrt{5}]$, 2 is irreducible. $2 \neq \pm 1$.

If $2 = \alpha\beta$, $4 = N(2) = N(\alpha\beta) = N(\alpha)N(\beta)$

$$\begin{array}{|c|} \hline 4 \times 1 \\ \hline 2 \times 2 \\ \hline 1 \times 4 \\ \hline \end{array}$$

β is a unit.
 α unit.

$$N(a+b\sqrt{5}) = a^2 + 5b^2 \in \{0, 1, 2, 3, \dots\}$$

$$N(2) = 4$$

If $N(\alpha) = N(a+b\sqrt{5}) = 1$ then $(a,b) = (\pm 1, 0)$, $\alpha = \pm 1$ is a unit.

So 2 is irreducible in $\mathbb{Z}[\sqrt{5}]$.

4 is reducible. $4 = 2 \times 2$

5 is reducible. $5 = (\sqrt{5})(-\sqrt{5})$

6 is reducible: $6 = 2 \times 3 = (1+\sqrt{5})(1-\sqrt{5})$ where all 2, 3, $1 \pm \sqrt{5}$ are irreducible by proof similar to above.

$\mathbb{Z}[\sqrt{5}]$ does not have unique factorization of elements.

But: ideals in \mathcal{O}_K (K any alg. number field) always have unique factorization.

Aside

Proof(?) of FLT for exponent 3, say: If x, y, z positive integers with $x^3 + y^3 = z^3$

$$(x+y)(x+wy)(x+w^2y) = z^3, \quad w = e^{2\pi i/3} = \frac{-1+\sqrt{3}}{2} \text{ is an algebraic integer}$$

$\mathbb{Z}[w]$ has unique factorization

In $\mathbb{Z}[\sqrt{5}]$, we don't have unique factorization of elements but we do have unique factorization of ideals.

$$(a) = \{ra : r \in \mathbb{Z}[\sqrt{5}]\}$$

$$(a,b) = \{ra + sb : r,s \in \mathbb{Z}[\sqrt{5}]\}$$

etc.

$$(6) = (2)(3) = (1+\sqrt{5})(1-\sqrt{5}) = \mathfrak{g}^2 \mathfrak{g} \bar{\mathfrak{g}}$$

not prime factors

where $\mathfrak{g}, \mathfrak{g}, \bar{\mathfrak{g}} \subset \mathbb{Z}[\sqrt{5}]$ are prime ideals. (not principal)

$$(2) = \mathfrak{g}^2$$

$$(3) = \mathfrak{g} \bar{\mathfrak{g}}$$

$$(1+\sqrt{5}) = \mathfrak{g} \mathfrak{g}$$

$$(1-\sqrt{5}) = \mathfrak{g} \bar{\mathfrak{g}}$$

$$\mathfrak{g} = (2, 1+\sqrt{5})$$

$$\mathfrak{g} = (3, 1+\sqrt{5})$$

$$\bar{\mathfrak{g}} = (3, 1-\sqrt{5})$$

If $A, B \subseteq R$ are ideals then $A+B, A \cap B, AB \subseteq R$ are ideals.

$$A+B = \{\alpha + \beta : \alpha \in A, \beta \in B\}$$

AB is not simply $\{\alpha\beta : \alpha \in A, \beta \in B\}$ is not an ideal in general because it's not closed under \pm .

$AB =$ closure of $\{\alpha\beta : \alpha \in A, \beta \in B\}$ under \pm .

$$= \{r_1\alpha_1\beta_1 + r_2\alpha_2\beta_2 + \dots + r_k\alpha_k\beta_k : k \geq 1, r_i \in R, \alpha_i \in A, \beta_i \in B\}$$

eg. $\mathfrak{g} = (2, 1+\sqrt{5}) \subset \mathbb{Z}[\sqrt{5}]$
 $= \{2r + (1+\sqrt{5})s : r, s \in \mathbb{Z}[\sqrt{5}]\}$

$$\mathfrak{g}^2 = (4, 2(1+\sqrt{5}), (1+\sqrt{5})^2) \subseteq (2)$$

\uparrow
 $-4+2\sqrt{5}$

$$2 = \underbrace{(1+\sqrt{5})(1-\sqrt{5})}_{\mathfrak{g}^1} + \underbrace{2(2)}_{\mathfrak{g}^2} \in \mathfrak{g}^2 \Rightarrow (2) \subseteq \mathfrak{g}^2$$

$$(2) = \mathfrak{g}^2$$

$$\bar{\mathfrak{g}} = (2, \frac{1-\sqrt{5}}{2-\sqrt{5}}) = \mathfrak{g}$$

$$\mathfrak{g} = (3, 1+\sqrt{5})$$

$$\bar{\mathfrak{g}} = (3, 1-\sqrt{5})$$

$$\mathfrak{g}\bar{\mathfrak{g}} = (9, 3(1-\sqrt{5}), 3(1+\sqrt{5}), 6) \subseteq (3)$$

$$3 = \underbrace{3}_{\mathfrak{g}} \cdot \underbrace{3}_{\bar{\mathfrak{g}}} - \underbrace{(1+\sqrt{5})}_{\mathfrak{g}} \cdot \underbrace{(1-\sqrt{5})}_{\bar{\mathfrak{g}}} \in \mathfrak{g}\bar{\mathfrak{g}} \Rightarrow (3) \subseteq \mathfrak{g}\bar{\mathfrak{g}}$$

$$\Rightarrow (3) = \mathfrak{g}\bar{\mathfrak{g}}$$

$$1+\sqrt{5} = \underbrace{(-2)}_{\mathfrak{g}} \cdot \underbrace{(1+\sqrt{5})}_{\mathfrak{g}} + \underbrace{(1+\sqrt{5})}_{\mathfrak{g}} \cdot \underbrace{3}_{\bar{\mathfrak{g}}} \in \mathfrak{g}\bar{\mathfrak{g}}$$

$$(1+\sqrt{5}) \in \mathfrak{g}\bar{\mathfrak{g}}$$

$$\mathfrak{g}\bar{\mathfrak{g}} = (2, 1+\sqrt{5})(3, 1+\sqrt{5}) = (6, 2(1+\sqrt{5}), 3(1+\sqrt{5}), (1+\sqrt{5})^2)$$

$$\underbrace{(1+\sqrt{5})(1+\sqrt{5})}_{(1+\sqrt{5})} \subseteq (1+\sqrt{5})$$

$$\mathfrak{g}\bar{\mathfrak{g}} = (1+\sqrt{5})$$

$$\mathfrak{g}\bar{\mathfrak{g}} = \bar{\mathfrak{g}} = (1-\sqrt{5})$$

$$\mathfrak{g} = (2, 1+\sqrt{5})$$

$$\bar{\mathfrak{g}} = (3, 1+\sqrt{5})$$

In \mathbb{Z} , "prime" usually means "positive irreducible". 2, 3, 5, 7, 11, 13, ...

(Irreducibles are $\pm 2, \pm 3, \pm 5, \pm 7, \pm 11, \dots$)

In alg. no. theory we refer to prime ideals and irreducible elements.

Let K be a ^(a/b) no. field i.e. a finite extension $K \supseteq \mathbb{Q}$, $n = [K:\mathbb{Q}] = \text{degree of the extension} < \infty$

$$\mathcal{O} = \{ \text{alg. integers in } K \}$$

$$K = \{ r_1 \alpha_1 + \dots + r_n \alpha_n : r_i \in \mathbb{Q} \}$$

$$\mathcal{O} = \{ r_1 \alpha_1 + \dots + r_n \alpha_n : r_i \in \mathbb{Z} \}$$



How are prime ideals in \mathbb{Z} i.e. $(p) \subset \mathbb{Z}$ related to prime ideals $\mathfrak{p} \subset \mathcal{O}$?

Every prime ideal $\mathfrak{p} \subset \mathcal{O}$ contains a unique prime ideal $(p) \subset \mathbb{Z}$

$$\mathfrak{p} \cap \mathbb{Z} \subset \mathbb{Z} \text{ prime}$$

$$\mathfrak{p} \cap \mathbb{Z} = (p) \subset \mathbb{Z}, \text{ } p \text{ ordinary prime (rational prime)}$$

In other direction, given a prime ideal $(p) \subset \mathbb{Z}$, this gives an ideal $(p) \subset \mathcal{O}$ may or may not be prime.

\mathcal{O} has unique factorization of ideals but not necessarily of elements.

An ideal $\mathfrak{p} \subset \mathcal{O}$ is prime if $a, b \in \mathcal{O}$, $ab \in \mathfrak{p} \Rightarrow a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.

Nonzero prime ideals $\mathfrak{p} \subset \mathcal{O}$ are maximal.

If R is a commutative ring with identity and $A \subset R$ is an ideal, A is maximal if $A \subset R$ (proper containment, $A \neq R$) and there is no ideal B with $A \subset B \subset R$.

$\Leftrightarrow R/A$ is a field.

A is prime if $a, b \in R$, $ab \in A \Rightarrow a \in A$ or $b \in A$

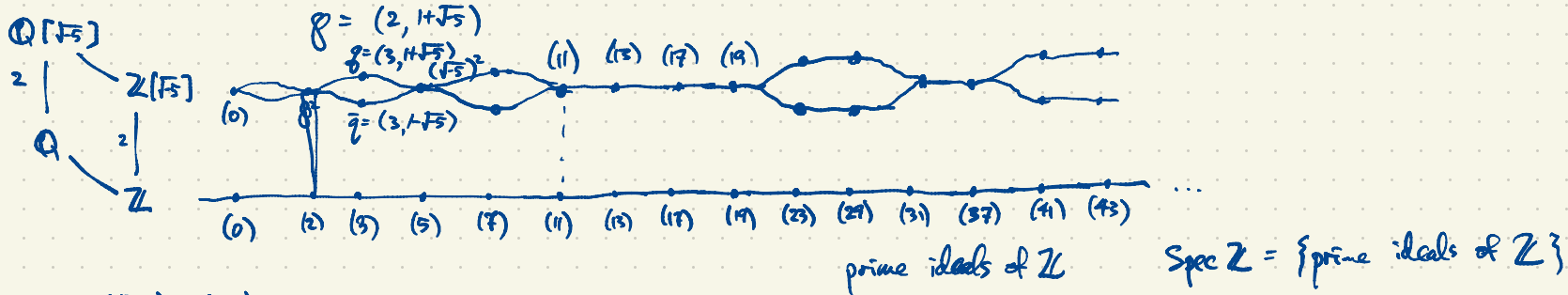
$\Leftrightarrow R/A$ is an integral domain. (no zero divisors)

Maximal \Rightarrow Prime.

Usually the converse is very far from true e.g. in $R = \mathbb{Q}[x, y]$ the ideal $(x) \subset R$ is prime. $R/(x) \cong \mathbb{Q}[y]$ is an integral domain. But $\mathbb{Q}[y]$ is not a field so $(x) \subset R$ is not maximal.

$(x) \subset (x, y) \subset R$ where $(x, y) = \{ f(x, y) \in R : f(0, 0) = 0 \} \subset R$ is maximal $R/(x, y) \cong \mathbb{Q}$ is a field.

In \mathcal{O} , if $A \subset \mathcal{O}$ is a nonzero ideal then A maximal iff A is prime. ("Nonzero primes are maximal.")



$$5 = (\sqrt{5})(-\sqrt{5})$$

To figure out how rational primes p behave in an extension:
 they either ramify (2, 5)
 or split
 or remain prime.

Think about what happens to a poly. ring like $\mathbb{R}[x]$ in a quadratic extension.

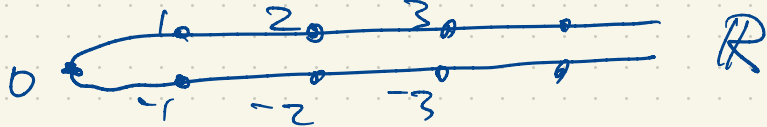
- Take $f(x) \in \mathbb{R}[x]$ a poly. of degree 2 and consider $\mathbb{R}[x]/(f(x))$.
- (i) If $f(x) \in \mathbb{R}[x]$ is irreducible eg. x^2+1 then $(f(x)) \subset \mathbb{R}[x]$ is maximal and $\mathbb{R}[x]/(f(x)) \cong \mathbb{C}$
 - (ii) If $f(x) \in \mathbb{R}[x]$ has two distinct real roots, say $f(x) = (x-a)(x-b)$, $a \neq b$, then

$$\mathbb{R}[x]/((x-a)(x-b)) = \mathbb{R}[x]/(x-a) \cap (x-b) \cong \mathbb{R}[x]/(x-a) \oplus \mathbb{R}[x]/(x-b) \cong \mathbb{R} \oplus \mathbb{R}$$
 - (iii) If $f(x) \in \mathbb{R}[x]$ has a double real root eg. $f(x) = x^2$ or $(x-a)^2$

$$\mathbb{R}[x]/(x^2) = \{a+bx \mid a, b \in \mathbb{R}\}$$

is the ring of dual numbers over \mathbb{R}
- (i): $(f(x)) \subset \mathbb{R}[x]$ is maximal (remains prime)
 (ii): $f(x) = (x-a)(x-b)$ splits $(x-a), (x-b) \subset \mathbb{R}[x]$ prime.
 (iii): $f(x) = (x-a)^2$ ramifies
- $(a+b\varepsilon)(c+d\varepsilon) = ac + (ad+bc)\varepsilon$

Squaring $\mathbb{R} \rightarrow \mathbb{R}$
 $x \mapsto x^2$



$\mathbb{Z}[x] \rightarrow \mathbb{Z}[\sqrt{5}]$
 $f(x) \mapsto f(\sqrt{5})$

$\mapsto \mathbb{Z}[x]/(x^2+5)$

In $\mathcal{O} = \mathbb{Z}[\sqrt{5}]$, what happens to an ordinary prime $p \in \mathbb{Z}$ in this extension? We should look at

$$\mathcal{O}/p\mathcal{O} \cong \frac{\mathbb{Z}[x]/(x^2+5)}{p\mathbb{Z}[x]/(x^2+5)} \cong \frac{\mathbb{Z}[x]}{(p, x^2+5)} \cong \mathbb{F}_p[x]/(x^2+5)$$

where $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} = \{0, 1, \dots, p-1\}$ field of order p .

use the isomorphism theorems for ring theory

Try a few primes $p \in \mathbb{Z}$

$$\mathcal{O}/2\mathcal{O} \cong \mathbb{F}_2[x]/(x^2+5) = \mathbb{F}_2[x]/(x^2+1)$$

$x^2+1 \in \mathbb{F}_2[x]$ **ramifies**: $x^2+1 = (x+1)^2$

So the rational prime $2 \in \mathbb{Z}$ ramifies in \mathcal{O} as $(2) = 2\mathcal{O} = \mathfrak{p}^2$ for some prime ideal $\mathfrak{p} \subset \mathcal{O}$.

(From before: $\mathfrak{p} = (2, 1+\sqrt{5})$)

$x^2+2 = x^2-1 = (x+1)(x-1)$ has two distinct roots in \mathbb{F}_3 .

So $(3) = 3\mathcal{O} = \mathfrak{q}\mathfrak{q}'$ where $\mathfrak{q} \neq \mathfrak{q}'$ are distinct prime ideals in \mathcal{O} .

(From before $\mathfrak{q} = (3, 1+\sqrt{5})$, $\mathfrak{q}' = \bar{\mathfrak{q}} = (3, 1-\sqrt{5})$)

We say (3) **splits** in \mathcal{O} . 3 is an irreducible element in \mathcal{O} but the ideal splits.

$$\mathcal{O}/3\mathcal{O} \cong \mathbb{F}_3[x]/(x^2+5) \cong \mathbb{F}_3[x]/(x^2+2) \cong \mathbb{F}_3[x]/(x^2-1)$$

$$\mathcal{O}/3\mathcal{O} \cong \mathbb{F}_3 \oplus \mathbb{F}_3$$

$$\mathcal{O}/5\mathcal{O} = \mathbb{F}_5[x]/(x^2+5) \cong \mathbb{F}_5[x]/(x^2) \cong \mathbb{F}_5[\varepsilon]$$

dual numbers

$x^2 = x \cdot x$

so $(5) = 5\mathcal{O} = (\sqrt{5})^2$

ramifies

$5 = (\sqrt{5})(-\sqrt{5})$

$(5) = (\sqrt{5})^2$

since $1 \in \mathcal{O}^*$

$\mathbb{Q}/7\mathbb{Q} = \mathbb{F}_7[x]/(x^2+5) = \mathbb{F}_7[x]/(x^2-2) \cong \mathbb{F}_7 \oplus \mathbb{F}_7$
 $x^2-2 = (x-3)(x-4)$
 (x^2-2) splits

the ideal $(7) = \mathcal{R}\mathcal{R}'$
 $\mathcal{R} \neq \mathcal{R}'$ distinct prime ideals in \mathbb{Q}

$\mathbb{Q}/11\mathbb{Q} \cong \mathbb{F}_{11}[x]/(x^2+5) = \mathbb{F}_{11}[x]/(x^2-6) \cong \mathbb{F}_{121}$

$\mathbb{O} = \mathbb{Z}[\sqrt{5}] = \{a+b\sqrt{5} : a, b \in \mathbb{Z}\}$

In \mathbb{F}_{11}

a	a ²
0	0
±1	1
±2	4
±3	9
±4	5
±5	3

$x^2-6 \in \mathbb{F}_{11}[x]$ is irreducible
 the ideal $(x^2-6) \subset \mathbb{F}_{11}[x]$ is maximal
 so it's prime.

$\mathbb{Q}/13\mathbb{Q} \cong \mathbb{F}_{13}[x]/(x^2+5) = \mathbb{F}_{13}[x]/(x^2-8) \cong \mathbb{F}_{169}$

In \mathbb{F}_{13}

a	a ²
0	0
±1	1
±2	4
±3	9
±4	3
±5	12
±6	10

$(11) = 11\mathbb{O} \subset \mathbb{O}$ is prime
 11 remains prime in \mathbb{O} .

$(13) = 13\mathbb{O} \subset \mathbb{O}$ is prime

$\mathbb{Q}/17\mathbb{Q} \cong \mathbb{F}_{17}[x]/(x^2+5) = \mathbb{F}_{17}[x]/(x^2-12) \cong \mathbb{F}_{289}$

In \mathbb{F}_{17} the squares are 0, 1, 4, 9, 16, 8, 2, 15, 13.

13 remains prime in \mathbb{O}
 $x^2-12 \in \mathbb{F}_{17}[x]$ irreducible
 $(x^2-12) \subset \mathbb{F}_{17}[x]$ is maximal hence prime

$\mathbb{Q}/19\mathbb{Q} \cong \mathbb{F}_{19}[x]/(x^2+5) \cong \mathbb{F}_{361} = \mathbb{F}_{19^2} = \mathbb{F}_{19}[\sqrt{5}] = \{a+b\sqrt{5} : a, b \in \mathbb{F}_{19}\}$

$\mathbb{O} = (19) \subset \mathbb{O}$ is maximal i.e. prime
 $(23) = \mathcal{S}\mathcal{S}'$, $\mathcal{S} \neq \mathcal{S}'$ prime ideals in \mathbb{O} .

$x^2+5 = (x-15)(x-8)$ in $\mathbb{F}_{23}[x]$

$\mathbb{Q}/23\mathbb{Q} \cong \mathbb{F}_{23}[x]/(x^2+5) = \mathbb{F}_{23}[x]/((x-15)(x-8)) \cong \mathbb{F}_5[x]/(x-15) \oplus \mathbb{F}_{23}[x]/(x-8)$

$-2 = 25 = 5^2$
 $5 = -18 = -2 \cdot 9 = 5^2 \cdot 3^2 = 15^2$

Theorem In $\mathbb{O} = \mathbb{Z}[\sqrt{5}]$, $p \in \mathbb{Z}$ ordinary prime, $(p) = \mathcal{P}\mathcal{O}$

$\left\{ \begin{array}{l} \text{ramifies iff } p \in \{2, 5\} \\ \text{splits iff } p \equiv 1, 3, 7 \text{ or } 9 \pmod{20} \\ \text{remain prime iff } p \equiv 11, 13, 17, \text{ or } 19 \pmod{20}. \end{array} \right.$

To prove this, it suffices to see how x^2+5 factors in $\mathbb{F}_p[x]$.

i.e. whether or not -5 is a square in $\mathbb{F}_p = \{0, 1, 2, \dots, p-1\}$.

$$\left(\frac{-5}{p}\right) = \begin{cases} +1 & \text{if } -5 \text{ is a square in } \mathbb{F}_p \\ -1 & \text{if } -5 \text{ is a nonsquare in } \mathbb{F}_p \\ 0 & \text{if } -5 = 0 \text{ in } \mathbb{F}_p \end{cases}$$

the Legendre symbol for odd primes p .

eg. $\left(\frac{-5}{17}\right) = -1$, $\left(\frac{-5}{23}\right) = +1$, $\left(\frac{-5}{5}\right) = 0$

Quadratic Reciprocity: If $p \neq q$ are distinct rational primes then $\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = \begin{cases} 1 & \text{if at least one of } p, q \text{ is } \equiv 1 \pmod{4} \\ -1 & \text{if } p \equiv q \equiv 3 \pmod{4} \end{cases}$

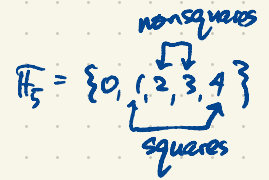
If $p \neq 2, 5$ then $\left(\frac{-5}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{5}{p}\right)$

$$\left(\frac{-1}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{4} \\ -1 & \text{if } p \equiv 3 \pmod{4} \end{cases}$$

$$\left(\frac{5}{p}\right)\left(\frac{p}{5}\right) = 1 \Rightarrow \left(\frac{5}{p}\right) = \left(\frac{p}{5}\right)$$

eg. $\left(\frac{5}{23}\right) = \left(\frac{23}{5}\right) = \left(\frac{3}{5}\right) = -1$

$$\left(\frac{5}{17}\right) = \left(\frac{17}{5}\right) = \left(\frac{2}{5}\right) = -1$$



$$\left(\frac{-5}{23}\right) = \left(\frac{-1}{23}\right)\left(\frac{5}{23}\right) = (-1)(-1) = 1 \Rightarrow -5 \text{ is a square in } \mathbb{F}_{23} \quad (15^2 = 8^2 = -5)$$

$$\left(\frac{-5}{17}\right) = \left(\frac{-1}{17}\right)\left(\frac{5}{17}\right) = (-1)(-1) = -1 \Rightarrow -5 \text{ is a nonsquare in } \mathbb{F}_{17}$$

$$\mathcal{O} \cap \mathbb{Z} = (\mathfrak{p}) \subset \mathbb{Z} \text{ prime}$$

The nonzero prime ideals in $\mathcal{O} = \mathbb{Z}[\sqrt{5}]$ are

- $\mathfrak{g} = (2, 1+\sqrt{5})$, $\mathfrak{g}' = (3, 1+\sqrt{5})$ (lying above $2, 3 \in \mathbb{Z}$)
- $(p) = p\mathcal{O}$, $p \in \mathbb{Z}$ ordinary prime of the form $p = 20k+11, 20k+13, 20k+17, 20k+19$
- prime ideals $\mathfrak{p} \subset \mathcal{O}$ lying above rational primes $p \in \mathbb{Z}$, $p \equiv 1, 3, 7 \text{ or } 9 \pmod{20}$.
i.e. $p \in \{7, 11, 13, 17, 19, 31, \dots\}$

Next: Which primes ramify in an extension?

All primes dividing the discriminant.

The extension $K \supseteq \mathbb{Q}[\sqrt{5}]$ (or $\mathbb{Z}[\sqrt{5}] \supseteq \mathbb{Z}$) has discriminant -20 . We'll talk about this next.

A lattice in \mathbb{R}^n is $\{a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n : a_i \in \mathbb{Z}\}$ where $\{\alpha_1, \dots, \alpha_n\}$ is a basis of \mathbb{R}^n .

For every number field $K \supseteq \mathbb{Q}$ i.e. extension field of degree $n = [K:\mathbb{Q}] < \infty$

K has a basis $\alpha_1, \dots, \alpha_n$ such that $\mathcal{O} = \{\alpha \in K : \alpha \text{ is an alg. int}\} = \text{lattice generated by } \alpha_1, \dots, \alpha_n$.

eg. $K = \mathbb{Q}[\sqrt{5}]$ has basis $\{1, \sqrt{5}\}$

$$K = \{a + b\sqrt{5} : a, b \in \mathbb{Q}\}$$

$$\mathcal{O} = \{a + b\sqrt{5} : a, b \in \mathbb{Z}\} \text{ is a lattice in } \mathbb{Q}^2$$

The trace map $\text{tr}: K \rightarrow \mathbb{Q}$ of the extension is the map $\text{tr}(\alpha) = \alpha + \sigma(\alpha)$ where $\sigma(a + b\sqrt{5}) = \overline{a + b\sqrt{5}} = a - b\sqrt{5}$
 $\sigma \in \text{Aut } K = \{1, \sigma\}$

The norm map $N: K \rightarrow \mathbb{Q}$ of the extension is

$$N(\alpha) = \alpha \sigma(\alpha) = \alpha \bar{\alpha}$$

$$N(\alpha\beta) = N(\alpha)N(\beta)$$

$$1(\alpha) = \alpha$$

$$\text{tr}(\alpha + \beta) = \text{tr}(\alpha) + \text{tr}(\beta)$$

The discriminant is the determinant of the $n \times n$ matrix whose (i, j) -entry is $\text{tr}(\alpha_i \alpha_j)$.

For $K = \mathbb{Q}[\sqrt{5}]$ with basis $\{1, \sqrt{5}\}$,
 α_1 α_2

$$\text{disc}_{\mathbb{Q}} K = \det \begin{bmatrix} \text{tr}(1 \cdot 1) & \text{tr}(1 \cdot \sqrt{5}) \\ \text{tr}(\sqrt{5} \cdot 1) & \text{tr}(\sqrt{5} \cdot \sqrt{5}) \end{bmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & -10 \end{vmatrix} = -20$$

$$\text{tr } 1 = 1 + \bar{1} = 2$$

$$\text{tr } \sqrt{5} = \sqrt{5} + (-\sqrt{5}) = 0$$

$$\text{tr } (-5) = -5 + \overline{-5} = -10$$

The rational primes dividing -20 are $2, 5$.

These are the ordinary primes that ramify in the extension.

For every nonzero ideal $A \subseteq \mathcal{O}$ we can think of A as a sublattice of \mathcal{O} with only finitely many cosets. $|\mathcal{O}/A| = \text{number of cosets} = \text{norm of the ideal } A \subseteq \mathcal{O}$.

$\mathcal{O}/A = \{ \underline{v+A} : v \in \mathcal{O} \}$ quotient of additive abelian groups i.e. quotient of \mathbb{Z} -modules

$$v+A = \{v+\pi : \pi \in A\}$$

If $\alpha \in \mathcal{O}$ then $(\alpha) = \alpha\mathcal{O} \subseteq \mathcal{O}$ and $N((\alpha)) = |N(\alpha)|$ (absolute value)

$$N(AB) = N(A)N(B)$$

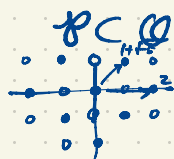
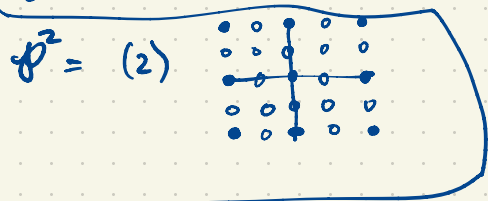
$$|\mathcal{O}/AB| = |\mathcal{O}/A| \cdot |\mathcal{O}/B|$$

This may be obvious if A, B rel. prime

eg. $\mathfrak{p} = (2, 1+\sqrt{5})$ has norm 2

$$\mathcal{O} = \{a+b\sqrt{5} : a, b \in \mathbb{Z}\}$$

i.e. $A+B = \mathcal{O}$



$\mathfrak{p} \subset \mathcal{O}$ is generated by $\begin{matrix} 2 \\ (2,0) \end{matrix}, \begin{matrix} 1+\sqrt{5} \\ (1,1) \end{matrix}$

\mathfrak{p} : •

\mathcal{O} : •, • two cosets of \mathfrak{p}

$$\mathfrak{p} = \{a+b\sqrt{5} : a, b \in \mathbb{Z}, a+b \text{ even}\}$$

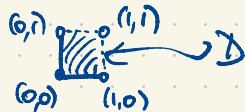
To prove $\mathfrak{p}^2 = (2)$ we can argue as before $\mathfrak{p}^2 \subseteq (2)$ and $(2) \subseteq \mathfrak{p}^2$.

$$N(\mathfrak{p}) = |\mathcal{O}/\mathfrak{p}| = 2$$

$$\mathcal{O} = \mathfrak{p} \sqcup (1+\mathfrak{p})$$

slightly tricky

\mathcal{O} has fund. domain $[0,1) \times [0,1)$



$$\text{Or: } N(\mathfrak{p}^2) = N(\mathfrak{p})^2 = 2^2 = 4$$

$$N((2)) = |N(2)| = |2 \cdot \bar{2}| = 4$$

$$\mathbb{R}^2 = \bigsqcup_{v \in \mathcal{O}} (v + D)$$

$\mathcal{O} = \mathbb{Z}[\sqrt{5}]$ has two kinds of nonzero ideals:

- (principal) (11), (13), (17), (19), $(7) = \mathfrak{p}\mathfrak{p}'$, $(2) = \mathfrak{p}^2, \dots$
 $(\sqrt{5})$, $(6) = \mathfrak{p}\mathfrak{p}\bar{\mathfrak{p}}$, $(3) = \mathfrak{q}\bar{\mathfrak{q}}$

prime

- (nonprincipal) $\mathfrak{g} = (2, 1+\sqrt{5})$, $\mathfrak{q} = (3, 1+\sqrt{5})$, \mathfrak{p}, \dots , $\mathfrak{p}^3 = 2\mathfrak{g}$

prime

non-prime

	principal	non-principal
principal	principal	non-principal
non-principal	non-principal	principal

is a (multiplicative) group of order 2. $(a)(b) = (ab)$

For every number field $K \supseteq \mathbb{Q}$ i.e. finite extension field

we have the ring of integers $\mathcal{O} \supseteq \mathbb{Z}$ in this extension
 the nonzero (fractional) ideals of \mathcal{O} form a mult. group and the principal ideals form a subgroup.

The quotient group $\{\text{ideals}\} / \{\text{principal ideals}\}$ (the ideal class group of the extension $K \supseteq \mathbb{Q}$) is a finite group.

The order of this group is denoted $h = h_K = |\{\text{ideals}\} / \{\text{principal ideals}\}|$ is the class number or ideal class number (= number of ideal classes = number of cosets of principal ideals) of the extension $K \supseteq \mathbb{Q}$.

$$h_{\mathbb{Q}[\sqrt{5}]} = 2$$

$$h_{\mathbb{Q}} = 1 \quad (\text{every nonzero ideal in } \mathbb{Z} \text{ is principal}).$$

$$h_{\mathbb{Q}(i)} = 1$$

Consequence of Minkowski's Theorem

$h_K = 1$ iff \mathcal{O} has unique factorization (i.e. of elements).

(ideals always have unique factorization)

Dedekind zeta function $\zeta_K(s)$ of a number field K .

eg. $\zeta_{\mathbb{Q}}(s) = \zeta(s) =$ Riemann zeta function

$$N(\mathcal{A}) = |\mathcal{O}/\mathcal{A}|$$

$$s \in \mathbb{C}, \quad \zeta_K(s) = \sum_{(0) \neq \mathcal{A} \subseteq \mathcal{O}} \frac{1}{N(\mathcal{A})^s}$$

Eg. Every ideal $\mathcal{A} \subseteq \mathbb{Z}$ is principal, $\mathcal{A} = n\mathbb{Z} = (n)$ for some $n \in \mathbb{Z}$, $(-n) = (n)$

$$N((0)) = |\mathbb{Z}/(0)| = |\mathbb{Z}| = \infty$$

$$N((n)) = |\mathbb{Z}/(n)| = n.$$

$$\zeta(s) = \zeta_{\mathbb{Q}}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots \quad \zeta(2) = \frac{\pi^2}{6}$$

$\zeta(s)$ is analytic on $\mathbb{C} - \{1\}$. (for $\operatorname{Re} s > 1$)
(pole at $s=1$)

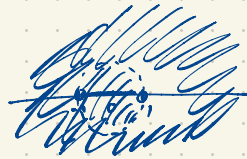
Compare: $f(s) = 1 + s + s^2 + s^3 + \dots$, convergent for $|s| < 1$.

This extends to the entire plane \mathbb{C} as a meromorphic function with a simple pole at $s=1$.

$$f(s) = \frac{1}{1-s}$$

In number theory we consider Dirichlet series $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$, $a_n \in \mathbb{R}$.
If it converges at some point $P \in \mathbb{C}$ then it converges "to the right".

The typical domain of convergence of a Dirichlet series is a half-plane $\operatorname{Re} s > \sigma$.



The zeta function $\zeta_K(s)$ of a number field converges in $\text{Re } s > 1$. But $\zeta_K(s)$ continues analytically to \mathbb{C} with a simple pole at $s=1$.

"Dirichlet series" includes zeta functions and L-functions.

$$\zeta(2) = \frac{\pi^2}{6}$$

the Basel problem

The Riemann zeta function has an Euler factorization

$$\begin{aligned} \zeta(s) &= \prod_p \frac{1}{1 - \frac{1}{p^s}} = \frac{1}{1 - \frac{1}{2^s}} \cdot \frac{1}{1 - \frac{1}{3^s}} \cdot \frac{1}{1 - \frac{1}{5^s}} \cdot \frac{1}{1 - \frac{1}{7^s}} \cdot \frac{1}{1 - \frac{1}{11^s}} \cdot \frac{1}{1 - \frac{1}{13^s}} \times \dots \\ &= (1 + \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{8^s} + \frac{1}{16^s} + \dots) (1 + \frac{1}{3^s} + \frac{1}{9^s} + \dots) (1 + \frac{1}{5^s} + \dots) (1 + \frac{1}{7^s} + \dots) (1 + \frac{1}{11^s} + \dots) (1 + \frac{1}{13^s} + \dots) \\ &= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \frac{1}{8^s} + \dots \end{aligned}$$

Analytic form of the FTA = fundamental theorem of arithmetic

This works for every number field $K \supseteq \mathbb{Q}$:

$$\zeta_K(s) = \sum_{\mathfrak{a} \neq 0} \frac{1}{N(\mathfrak{a})^s} = \prod_{\mathfrak{p}} \left(\frac{1}{1 - \frac{1}{N(\mathfrak{p})^s}} \right) = \prod_{\mathfrak{p}} \frac{1}{1 - N(\mathfrak{p})^{-s}} = \prod_{\mathfrak{p}} (1 - N(\mathfrak{p})^{-s})^{-1}$$

since ideals in \mathcal{O} have unique factorization as products of prime ideals

$$N(\mathfrak{a}) = |\mathcal{O}/\mathfrak{a}|$$

\mathfrak{a} nonzero ideal

nonzero prime ideals $\mathfrak{p} \subset \mathcal{O}$

Eg. $\mathcal{O} = \mathbb{Z}[\sqrt{5}]$ has prime ideals:

Norm 2: $\mathfrak{p} = (2, 1 + \sqrt{5})$

Norm 3: $\mathfrak{p} = (3, 1 + \sqrt{5}), \bar{\mathfrak{p}} = (3, 1 - \sqrt{5})$

Norm 5: $(\sqrt{5})$

Norm 7: $\mathfrak{p}, \bar{\mathfrak{p}} = \bar{\mathfrak{p}}$

Norm 11^2 : (11)

$$\mathcal{O}/_{11\mathcal{O}} = \mathcal{O}/(11) \cong \mathbb{F}_{121}$$

Norm 31: $\mathfrak{p}, \bar{\mathfrak{p}} \quad \mathfrak{p}\bar{\mathfrak{p}} = (31) \quad N(\mathfrak{p}) = |\mathcal{O}/\mathfrak{p}| = 31$

$$\zeta_{\mathcal{O}[\sqrt{5}]}(s) = \frac{1}{1 - \frac{1}{2^s}} \left(\frac{1}{1 - \frac{1}{3^s}} \right)^2 \frac{1}{1 - \frac{1}{5^s}} \left(\frac{1}{1 - \frac{1}{7^s}} \right)^2 \frac{1}{1 - \frac{1}{11^s}} \frac{1}{1 - \frac{1}{13^s}} \frac{1}{1 - \frac{1}{17^s}} \frac{1}{1 - \frac{1}{19^s}} \left(\frac{1}{1 - \frac{1}{31^s}} \right)^2 \times \dots$$

Non-prime ideals: $(2) = \mathfrak{p}^2$ of norm 4
 $\mathfrak{p}\bar{\mathfrak{p}} = (1 + \sqrt{5})$ of norm 6
 $\mathfrak{p}\bar{\mathfrak{p}} = (1 - \sqrt{5})$ of norm 6

$$\zeta_{\mathbb{Q}[\sqrt{-5}]}(s) = \frac{1}{1-\frac{1}{2^s}} \left(\frac{1}{1-\frac{1}{3^s}} \right)^2 \frac{1}{1-\frac{1}{5^s}} \left(\frac{1}{1-\frac{1}{7^s}} \right)^2 \frac{1}{1-\frac{1}{11^s}} \frac{1}{1-\frac{1}{13^s}} \frac{1}{1-\frac{1}{17^s}} \frac{1}{1-\frac{1}{19^s}} \left(\frac{1}{1-\frac{1}{31^s}} \right)^2 \dots = \left(1 + \frac{1}{2^s} + \frac{1}{4^s} + \dots \right) \left(1 + \frac{2}{3^s} + \frac{3}{9^s} + \frac{4}{27^s} + \dots \right) \left(1 + \frac{1}{5^s} + \frac{1}{25^s} + \dots \right) \dots$$

Non-prime ideals: $(2) = \mathfrak{p}^2$ of norm 4
 $8\mathfrak{p} = (1+\sqrt{-5})$ of norm 6
 $8\bar{\mathfrak{p}} = (1-\sqrt{-5})$ of norm 6

$$= \sum_{0 \neq A \subseteq \mathbb{O}} \frac{1}{N(A)^s}$$

$$= 1 + \frac{1}{2^s} + \frac{2}{3^s} + \dots$$

But what else can we do with zeta functions?

\uparrow ideals of norm 3: $\mathfrak{q}, \bar{\mathfrak{q}}$

Eg. If m, n are randomly chosen large integers, what is the probability that m, n are relatively prime? About 61%.

$$\{ (m, n) \in [1, N]^2 : \gcd(m, n) = 1 \}$$

$$\lim_{N \rightarrow \infty} \frac{\text{Number of } (m, n) \text{ with } \gcd(m, n) = 1}{N^2} = \frac{6}{\pi^2} \approx 0.60793$$

Why? $\gcd(m, n) = 1 \iff$

- $\left\{ \begin{array}{l} m, n \text{ are not both divisible by } 2 \\ \text{and} \dots \dots \dots 3 \\ \dots \dots \dots 5 \\ \dots \dots \dots 7 \\ \dots \dots \dots 11 \\ \text{and} \dots \dots \dots \end{array} \right.$

with probability $1 - \frac{1}{4} = 1 - \frac{1}{2^2}$
 $1 - \frac{1}{9} = 1 - \frac{1}{3^2}$
 $1 - \frac{1}{25} = 1 - \frac{1}{5^2}$
 $1 - \frac{1}{49} = 1 - \frac{1}{7^2}$
 $1 - \frac{1}{121} = 1 - \frac{1}{11^2}$

with probability $(1 - \frac{1}{2^2})(1 - \frac{1}{3^2})(1 - \frac{1}{5^2})(1 - \frac{1}{7^2})(1 - \frac{1}{11^2}) \times \dots = \frac{1}{\zeta(2)}$

This value $\frac{1}{\zeta(2)} \approx 0.61$ is also the probability that a randomly chosen large integer n is squarefree. (not divisible by any prime more than once).

Given $m_1, m_2, \dots, m_k \in \mathbb{Z}$, we say m_1, \dots, m_k are relatively prime if $\gcd(m_1, m_2, \dots, m_k) = 1$. $\iff (m_1, m_2, \dots, m_k) = (1) = \mathbb{Z}$

$$m_1\mathbb{Z} + m_2\mathbb{Z} + \dots + m_k\mathbb{Z}$$

eg. 6, 10, 15 are relatively prime since $\gcd(6, 10, 15) = 1$.

Given three large integers, how often are they relatively prime? $\frac{1}{\zeta(3)}$.

eg. $K = \mathbb{Q}[\sqrt{5}]$, $\mathcal{O} = \mathbb{Z}[\sqrt{5}] = \{a + b\sqrt{5} : a, b \in \mathbb{Z}\}$.

Given $\alpha, \beta \in \mathcal{O}$ at random, how often are α, β relatively prime? i.e. $\gcd(\alpha, \beta) = 1$ (the only elements of the ring \mathcal{O} dividing both α, β are the units ± 1).

$$\iff (\alpha, \beta) = \mathcal{O} = (1)$$

The prob. that $\alpha, \beta \in \mathcal{O}$ are relatively prime is $\frac{1}{\zeta_K(2)}$

$$\zeta_K(s) = \frac{1}{1-\frac{1}{2^s}} \cdot \frac{1}{1-\frac{1}{5^s}} \cdot \prod_{\substack{p=1,3,7,9 \\ \text{mod } 20}} \left(\frac{1}{1-\frac{1}{p^s}} \right)^2 \cdot \prod_{\substack{p=11,13,17,19 \\ \text{mod } 20}} \left(\frac{1}{1-\frac{1}{p^{2s}}} \right)$$

$$\zeta_K(2) \approx 1.855557$$

$$\frac{1}{\zeta_K(2)} \approx 0.53892$$

2, 5 ramify

primes that split

p remains prime

About 54% of the time, $\alpha, \beta \in \mathcal{O}$ are relatively prime.

$$= \zeta_{\mathbb{Q}}(s) \cdot \frac{1}{1-\frac{1}{3^s}} \cdot \frac{1}{1-\frac{1}{7^s}} \cdot \frac{1}{1+\frac{1}{11^s}} \cdot \frac{1}{1+\frac{1}{13^s}} \cdot \frac{1}{1+\frac{1}{17^s}} \cdot \frac{1}{1+\frac{1}{19^s}} \cdot \frac{1}{1-\frac{1}{23^s}} \cdot \frac{1}{1-\frac{1}{29^s}} \cdot \frac{1}{1+\frac{1}{31^s}} \dots$$

Riemann zeta $\zeta_{\mathbb{Q}}(s)$