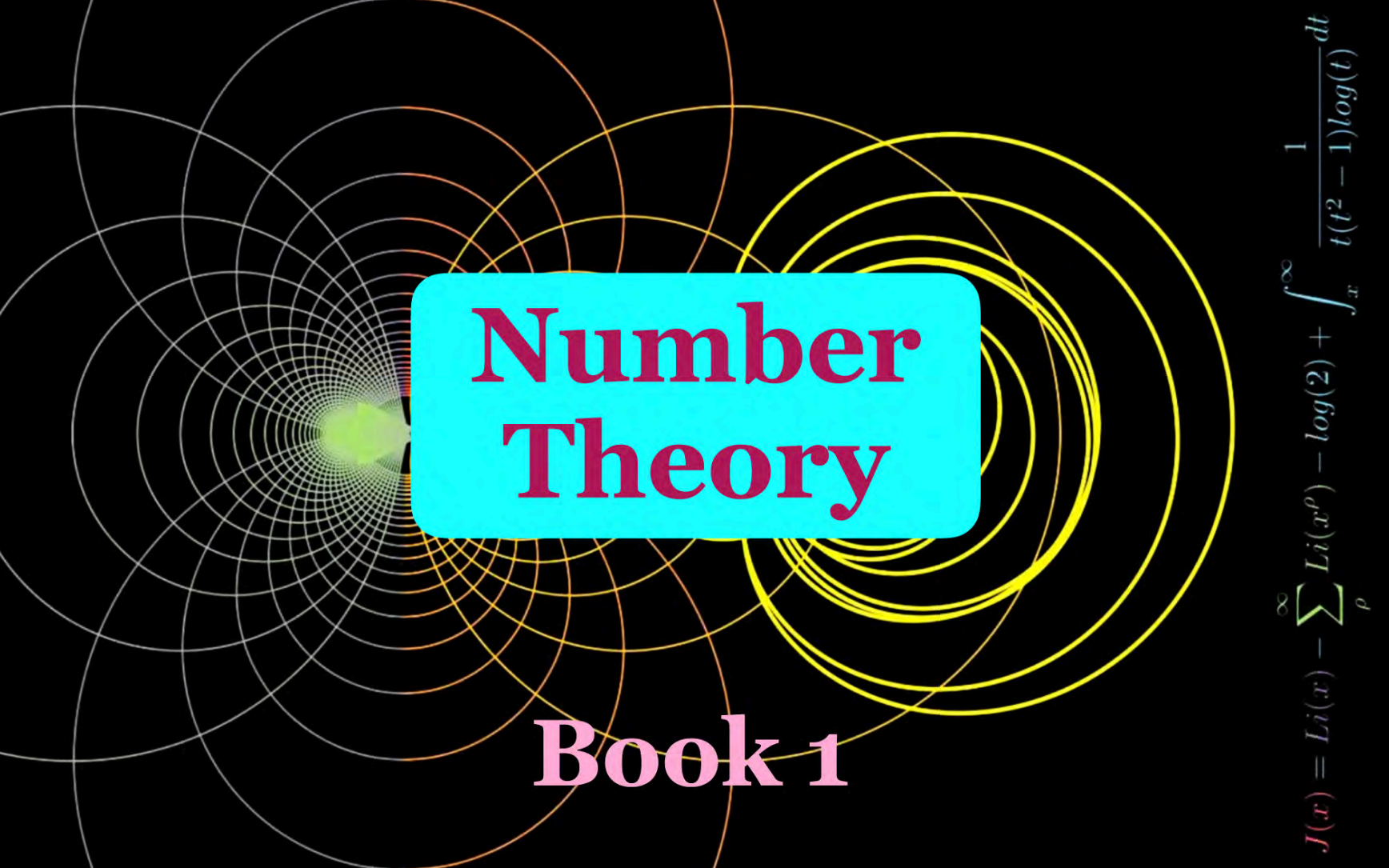

Number
Theory
Book 1
$J(x) = Li(x) - \sum_{\rho} Li(x^{\rho}) - \log(2) + \int_x^{\infty} \frac{1}{t(t^2-1)\log(t)} dt$

Rational integers ("ordinary integers") $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$

(Algebraic) integers: A number $\alpha \in \mathbb{C}$ is <u>algebraic</u> if it is a root of a nonzero poly. with coefficients in $\mathbb{Q}$

i.e. $f(\alpha) = 0$ for some $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, $a_i \in \mathbb{Q}$ i.e. $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, $a_i \in \mathbb{Z}$ not all zero.

eg. $\frac{\sqrt{2}}{7}$ is algebraic since it's a root of $x^2 - \frac{2}{49} \in \mathbb{Q}[x]$ or $49x^2 - 2 \in \mathbb{Z}[x]$.

We say α is an (<u>algebraic</u>) <u>integer</u> if α is a root of a <u>monic</u> poly. with coefficients in $\mathbb{Z}$ ie.

$f(\alpha) = 0$ for some $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, $a_i \in \mathbb{Z}$.

eg. $\sqrt{2}$ is integral (it's an (algebraic) integer)

If $\mathcal{O} = \{\alpha \in \mathbb{C} : \alpha \text{ is an algebraic integer}\} \subset \mathcal{A} \subset \mathbb{C}$

$\mathcal{A} = \{\alpha \in \mathbb{C} : \alpha \text{ is algebraic}\} \subset \mathbb{C}$

$\mathcal{O}$ is the <u>ring</u> of alg. int.

$\mathcal{A}$ is the <u>field</u> of alg. numbers.

$\mathcal{O} \cap \mathbb{Q} = \mathbb{Z}$ (the rational integers are the ordinary integers).

Pell's equation: $x^2 - 5y^2 = 1$ has infinitely many (ordinary) integer solutions.

x	y
± 1	0
± 9	± 4
± 161	± 72
± 2889	± 1292
$\vdots$	$\vdots$

$x^2 - 5y^2 = 1 \iff (x + y\sqrt{5})(x - y\sqrt{5}) = 1$ where $x + y\sqrt{5} \in \mathcal{O}$ has Norm is $N(x + y\sqrt{5}) = (x + y\sqrt{5})(x - y\sqrt{5}) = x^2 - 5y^2$

$N(\alpha\beta) = N(\alpha)N(\beta)$ so if $\alpha = 9 + 4\sqrt{5}$ then $N(\alpha) = 9^2 - 5 \cdot 4^2 = 1$ so $N(\alpha^k) = N(\alpha)^k = 1^k = 1$

eg. $(9 + 4\sqrt{5})^2 = 81 + 80 + 72\sqrt{5} = 161 + 72\sqrt{5}$ also has norm 1 so $161^2 - 5 \cdot 72^2 = 1$

$(9 + 4\sqrt{5})^3 = (161 + 72\sqrt{5})(9 + 4\sqrt{5}) = 2889 + 1292\sqrt{5}$

$\mathcal{A}$ is the fraction field of $\mathcal{O}$ ie. $\mathcal{A} = \{\frac{\alpha}{\beta} : \alpha, \beta \in \mathcal{O}, \beta \neq 0\}$

$\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}$.

eg. $\sqrt{2}, \sqrt{3} \in \mathcal{O}$ (root of $x^2 - 2$, $x^2 - 3$ espectively)

$\Rightarrow \sqrt{2} \cdot \sqrt{3} = \sqrt{6} \in \mathcal{O}$ (root of $x^2 - 6$)

$\sqrt{2} + \sqrt{3} \in \mathcal{O}$ (root of $x^4 - 10x^2 + 1$)

the min. poly. of $\sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$.

$x = \sqrt{2} + \sqrt{3}$

$x^2 = 5 + 2\sqrt{6}$

$x^2 - 5 = 2\sqrt{6}$

$x^4 - 10x^2 + 25 = 24$

$x^4 - 10x^2 + 1 = 0$

Let $\alpha \in \mathbb{C}$. If α is algebraic then it has a minimal poly. over $\mathbb{Q}$ i.e. $m(x) \in \mathbb{Q}[x]$ is monic with $m(\alpha) = 0$ and $\deg m(x)$ is as small as possible. In this case $m(x)$ is unique. All polynomials in $\mathbb{Q}[x]$ having α as a root are multiples of $m(x)$.

$$(m(x)) = \{ h(x)m(x) : h(x) \in \mathbb{Q}[x] \}$$

$\mathbb{Q}[x]$ is a principal ideal ring (every ideal is principal).

Review: Let R be a commutative ring with identity $1 \in R$. (eg. $\mathbb{Z}$, $\mathbb{Q}[x]$).

An ideal is a subset $J \subseteq R$, $0 \in J$ such that J is closed under R-linear combinations i.e. $ra + sb \in J$ for all $r, s \in R$, $a, b \in J$. (Every ideal is a subring but not conversely).

Given $a_1, \dots, a_k \in R$, these elements generate an ideal $(a_1, a_2, \dots, a_k) = \{ r_1a_1 + r_2a_2 + \cdots + r_ka_k : r_1, \dots, r_k \in R \} \subset R$.

eg. in $\mathbb{Z}$, $(21, 6) = \{ 21r + 6s : r, s \in \mathbb{Z} \} = (3)$

Every ideal in $\mathbb{Z}$ is principal ie. generated by a single element.

Return to previous setting $\alpha \in \mathbb{C}$ algebraic.

There is a ring homomorphism $\mathbb{Q}[x] \longrightarrow \mathbb{C}$, $f(x) \longmapsto f(\alpha)$ with kernel $J = \{ f(x) \in \mathbb{Q} : f(\alpha) = 0 \} \subset \mathbb{Q}[x]$ an ideal.

Since α is algebraic, $J = (m(x))$ with $m(x) \neq 0$. Scale $m(x)$ if needed to get $m(x)$ monic. Then $m(x)$ is unique; it's the minimal poly. of α over $\mathbb{Q}$.

$\mathbb{Z}[x]$ is not a principal ideal ring

$\mathbb{Q}[x, y]$ - - - - - - - - - -

In $\mathbb{Q}[x, y]$, $(x^2, xy, y^2) = \{ f(x, y) \in \mathbb{Q}[x, y]$ with no const. term, no x term, no y term$\}$ is a non-principal ideal. This ideal cannot be generated by 1 or 2 generators; you need at least 3 generators to generate it.

In $\mathbb{Q}[x, y]$, every ideal is finitely generated (there is a finite list of generators) (Hilbert's basis theorem)

In number theory, a number (algebraic) field is a finite extension $K \geq \mathbb{Q}$ eg. $\mathbb{Q}[\sqrt{2}] = \{ a + b\sqrt{2} : a, b \in \mathbb{Q} \} \supset \mathbb{Q}$ is a quadratic extension.

$[K:\mathbb{Q}]$ = degree of the extension

$[E:F]$ = degree of field extension $E \supseteq F$ (F subfield of E)
= dimension of E as a vector space over F

$[\mathbb{Q}[\sqrt{2}] : \mathbb{Q}] = 2$ since $\{1, \sqrt{2}\}$ is a basis for $\mathbb{Q}[\sqrt{2}]$ over $\mathbb{Q}$.

$\mathbb{C} \supset \mathbb{R} \supset \mathbb{Q}$

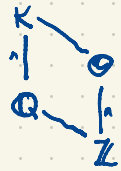
$[\mathbb{C}:\mathbb{R}] = 2$

$[\mathbb{C}:\mathbb{Q}] = \infty$

$[\mathbb{R}:\mathbb{Q}] = \infty$.

$K \supseteq \mathbb{Q}$ finite extension $n = [K:\mathbb{Q}] < \infty$. All elements $\alpha \in K$ are algebraic i.e. $K \subset \mathcal{A} = \{\text{algebraic numbers}\}$.

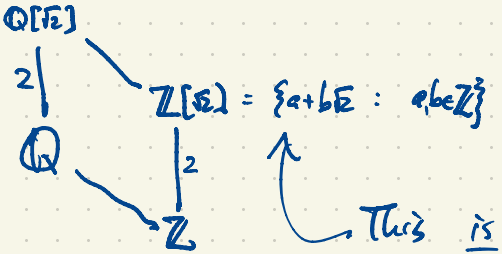
If $\alpha \in K$ then $1, \alpha, \alpha^2, \cdots, \alpha^n$ so there exist $a_0, a_1, \cdots, a_n \in \mathbb{Q}$ not all zero, s.t. $a_0 + a_1\alpha + \cdots + a_n\alpha^n = 0$
so α is algebraic. (of degree $\leq n$. The degree of an algebraic number is the degree of its min. poly. $= [\mathbb{Q}(\alpha):\mathbb{Q}]$



$\mathcal{O} = \{\text{alg. integers in } K\}$

$K = \left\{\frac{\alpha}{\beta} : \alpha, \beta \in \mathcal{O},\ \beta \neq 0\right\}$.

eg.



$\mathcal{O} = \{r_1\alpha_1 + \cdots + r_n\alpha_n : r_i \in \mathbb{Z}\}$

$\alpha_1, \cdots, \alpha_n$ basis for $\mathcal{O} \supseteq \mathbb{Z}$

$\alpha_1, \cdots, \alpha_n$ basis for $K \supseteq \mathbb{Q}$

$\mathcal{O}$ is not (usually) always a principal ideal ring.

Every ideal $J \subset \mathcal{O}$ has the form

$J = (a)$ or (a, b)