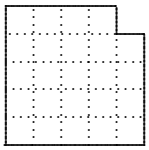
On Learning to Read and Write Proofs

Three Examples of Proofs

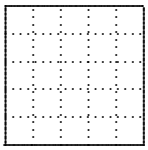
Mathematical proofs are exceptionally varied in length, style and difficulty. It is not possible to represent the full gamut of proof techniques and formats with three examples, but we have chosen these examples to illustrate some of this variety.

Our first example concerns tiling problems. Consider the following three regions:

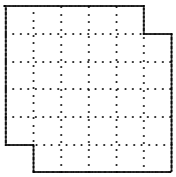
Region A



Region B



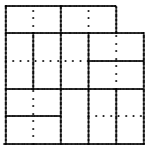
Region C



We are given the problem of determining which of these regions may be tiled using 1×2 tiles ('dominoes'). To tile a region using dominoes means to cover the region entirely using dominoes, without any overlapping tiles.

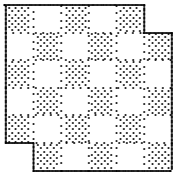
Theorem 1. Of the three regions shown above, region A can be tiled with 1×2 tiles ('dominoes') but regions B and C cannot.

Proof. There are many ways to tile region A, for example:



The fact that region B cannot be tiled using dominoes follows from the fact that it contains an odd number (25) of 1×1 cells. Since each domino covers 2 cells, any region that can be tiled with n dominoes must contain exactly $2n$ cells, and even number.

Although region C contains an even number (34) of 1×1 cells, nevertheless it cannot be tiled using dominoes. To see this, color the cells alternately gray and white as shown:



If the region could be tiled using dominoes, then every domino would cover one gray cell and one white cell; therefore n dominoes would cover exactly n gray cells and n white cells. Since there are different numbers of gray and white cells in the figure (18 and 16, respectively), no such tiling is possible. □

From the proof we see that in order for a region to be tilable using dominoes, it is necessary (but not sufficient) that it contain an even number of 1×1 cells.

Theorem 2. Let a, b, n be integers. If a and b are divisible by n, then so are $a + b$ and $a - b$.

Proof. Suppose a and b are divisible by n, say $a = rn$ and $b = sn$. Then both $a+b = (r+s)n$ and $a-b = (r-s)n$ are divisible by n, by definition. □

The *geometric mean* of two positive real numbers a and b is by definition $\sqrt{ab}$. The following result compares this quantity with the *arithmetic mean* (i.e. *average*) $(a + b)/2$. First consider a small example: the geometric mean of 4 and 6 is $\sqrt{24} \approx 4.8990$, whereas the arithmetic mean is 5. Both means lie between 4 and 6, but the geometric mean is typically lower than the arithmetic mean.

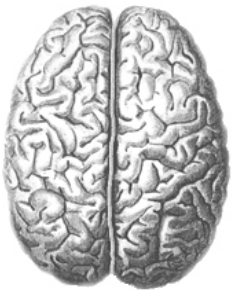
Theorem 3. Let a and b be positive real numbers. Then $\sqrt{ab} \leq \frac{a+b}{2}$.

Proof. Expanding $(a-b)^2 \geq 0$ gives $a^2 - 2ab + b^2 \geq 0$ which implies that $a^2 + 2ab + b^2 \geq 4ab$. Taking the square root of both sides yields $a + b \geq 2\sqrt{ab}$ which yields the desired result. □

The Art and Science of Writing Proofs

It should first be recognized that reading proofs and writing proofs are different skills. One might generally say that reading proofs is easier than writing proofs, but this statement requires some clarification. Reading a proof (and checking the details to make sure the proof is correct) is primarily a left-brain activity; whereas writing a proof typically involves

both sides of the brain. The roles of the two hemispheres of the brain are traditionally differentiated as shown:

Left Brain Activity		**Right Brain Activity**
sequential reasoning		parallel reasoning
deterministic approach		randomized approach
speech		imagination
case-by-case analysis		intuition
reductionist		holistic

It should be noted that attributing a particular brain activity exclusively to one physical side of the brain or the other is an oversimplification of the true complex nature of the brain. We have listed only some of the more commonly attributed functions of the two hemispheres. Moreover the popular literature on the subject contains a great deal of tommyrot (such as the notion that brain lateralization is also evident in the appearance of the left and right sides of a person's face). Regardless of the degree to which brain lateralization (the differentiation of roles of the two hemispheres) is physically responsible for the distinction between these two styles of brain activity, we find that the distinction is a very useful one for understanding the different tasks involved in mathematical reasoning.

For example, reading and understanding a proof is a rather sequential task—a typically left-brain activity. Writing a proof involves some of the same skills, but usually requires a great deal of intuition as well—very much a right-brain function. This point is overlooked by many amateur or even professional psychologists who typically categorize mathematical activity as left-brain activity. But then, many non-mathematicians view mathematicians as accountants.

Three Useful Analogies

Some insights into the nature of reading and writing proofs can be made by comparing this skill to the process of learning a language, or traversing a maze, or performing a household repair.

Learning a Language

Learning to write mathematical proofs is quite like learning any other language. The way you learned English was not by starting with a book on the complete rules of English grammar and pronunciation. And if you want to learn Russian as an adult, a book would still not necessarily be the most natural way to learn.

You learned English by listening to English being spoken. You imitated simple sounds, then words, and finally sentences. Then you began to build your own sentences: at first

by slight modifications of sentences you have heard before, and later building more complicated sentences based on rules that you have learned by intuition—rules that you have never read in any book. Occasionally you make a mistake, putting together invalid words or sentences, and someone older corrected you. This was of course part of the learning process. As years went by, you learned more and more about what makes the English language work. Then at some point in an English class you learned about subjects and predicates; nouns, verbs, adjectives, prepositions, etc. and you thought 'Wow, this is what I have been doing all these years intuitively!'.

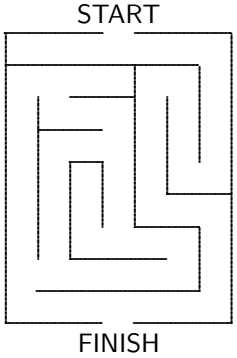
эътиқогларидан, миллий ёки ижтимоий келиб чиқишидан, мулкий аҳволи, қайси табақага мансублиги ва бошга ҳолатларидан қатъи назар, мазкур Декларайцияда эълон қулинган барча ҳуқуқлар ва эркинликлар соҳиби бўлиши керак. Бундан ташқари, инсон мансуб бўлган мамлакат ёки

You may decide as an adult to learn another spoken language, say Russian. As an adult, you probably won't learn Russian as easily as you learned your first spoken language as a child. But once again you will need to do much more than read a book. You will need to listen carefully to a proficient speaker, try to imitate them, accept criticism when your pronunciation and grammar are poor, correct yourself in response to the criticism and try again, etc. Learning as an adult, you will probably benefit somewhat from rules of grammar (at least understanding the difference between a noun and a verb, singular and plural, etc.) but most of your progress will be based on hard work, a strong motivation to succeed, a good teacher, and a positive attitude toward constructive criticism. With the appropriate translation from spoken language to mathematical language, these are also important factors in learning to read and write good proofs. Unless your name is Euclid of Alexandria, you will probably have a hard time learning all this on your own.

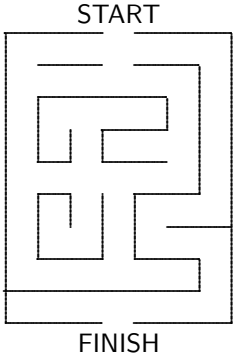
Traversing a Maze

Some proofs are constructed by a very routine (left-brain) process; others require much more creativity (right-brain activity). We explain this difference by comparing the task of proving a mathematical statement, to finding one's way through a maze. Consider the following four examples of mazes:

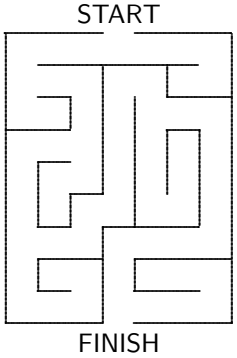
Maze A



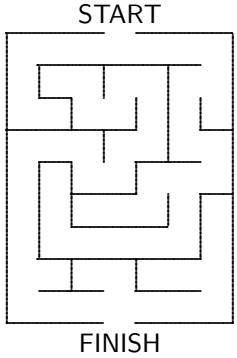
Maze B



Maze C



Maze D



Maze A can be traversed by the follow-your-nose strategy: simply keep walking. If you hit a wall, turn left or right but don't stop walking. And don't reverse your steps. At every step do the only reasonable thing. The proof of Theorem 2 above is of this variety: Here one starts by assuming the hypothesis that a and b are divisible by n. By invoking the definition of divisibility, this directly leads us to introduce new quantities r and s such that $a = rn$ and $b = sn$. This is clearly progress! since this gives expressions for a and b that can be substituted into $a \pm b$ to obtain $a \pm b = (r \pm s)n$ and the result follows. More generally any time a hypothesis leads to equations, you would do well to take advantage of this! since you are likely more familiar with equations than other mathematical statements; in particular you have experience with substituting values from one equation into another.

Traversing Maze B is not quite as easy; if one takes the wrong turn at the start then one gets lost. Here another strategy is helpful: work backwards from the Finish to the Start! This strategy can sometimes help in writing proofs: if you cannot see how to move forward from the hypothesis, try asking yourself whether you can reformulate the desired conclusion in a way that looks more accessible. But be careful: if you assume the conclusion and prove the hypothesis, this is actually the converse of what is required. Unless the steps are reversible then it won't give you a proof of the desired result. But maybe in your rough work, this kind of brainstorming might at least suggest something that could work in the forward direction. A good example of this approach is Theorem 3. How does one find the inspiration to start with the fact that $(a-b)^2 \geq 0$? This is suggested by working in reverse: if one wants to prove that

$$\sqrt{ab} \leq \frac{a+b}{2}$$

then by cross-multiplying and then squaring both sides (to get rid of the ugly square root) one sees that this is equivalent to

$$4ab \leq a^2 + 2ab + b^2.$$

This in turn is equivalent to

$$0 \leq a^2 - 2ab + b^2.$$

Factoring this expression, we see that really all we have to prove is that $0 \leq (a-b)^2$. But this is obviously true! Fortunately it is possible to start our proof with the statement $0 \leq (a-b)^2$ which is obviously true, and then by reversing these steps we obtain the desired result. The rough work amounts to starting at the finish line and working our way back to something we know; and we are lucky that all these steps are reversible.

Maze C is quite possible, but one can't expect to traverse it by the simple follow-your-nose approach. This takes either a little more insight or luck or trial and error. We might classify the proof of Theorem 1 in this category.

Maze D is impossible. Unfortunately the kind of mathematical problems that mathematicians attempt on a daily basis, are not always guaranteed to have solutions. If the follow-your-nose approach doesn't work, one may try for a solution involving either more insight or much hard work. Failing this, one may be forced to concede that a solution may be either too hard or nonexistent.

Performing a Household Repair

Imagine that you have no experience with repairing household items, and I have assigned you the task of fixing a table leg which is loose due to a loose screw. I give you a toolbox containing just two items: a screwdriver and a hammer. With some intelligence you will probably realize that the screwdriver exactly fits the loose screw; turning it counter-clockwise makes the leg even more loose, and turning it clockwise tightens the leg. Problem solved!

What if instead I had asked you to fix a leaking faucet? This is a more difficult job but this time I give you a much bigger selection of tools: various wrenches, plumber's putty and tape, a blowtorch, solder and flux, maybe a screwdriver and hammer and set of wood chisels. I assure you that you have all the necessary tools, and more. Without really knowing what you're doing, how likely is it that you will fix the leaking faucet? In this case having extra tools is not an asset; rather it makes the job more confusing unless you really know how each tool is used.

Given that you have very limited experience with writing your own proofs, and assuming that your instructor is a reasonable taskmaster (I try to be!) you can expect that any proofs you are asked to write at this stage will be do-able with the few tools you have available. In other words, the fact that you have been given very few tools to work with is an asset rather than a hindrance: you can expect to figure out in short order which tool fits, and how.

If you find yourself stuck at this early stage, perhaps the hammer doesn't seem to be helping to tighten the table leg, you should expect you're overlooking something obvious (like the screwdriver).

General Remarks on Proofs

We're not giving you a complete manual on how to write proofs. That book has never been written! Nor would it be that helpful to you at this early stage. But there are some general pointers that may be helpful in the learning process.

The Essence of a Proof

A *proof* of a statement is an explanation that convinces the reader that the statement is true. Every line of the proof is based only on previously known facts, such as axioms, hypotheses of the theorem (i.e. anything clearly stated as an assumption in the statement of the theorem, as a condition under which the conclusion is valid), previously proved theorems, facts demonstrated in preceding lines of the same proof, etc. A *theorem* is a statement for which a proof is available. Sometimes we instead call such a proven statement a *lemma* or a *proposition* (usually indicating that it leads to a more general result; often a proposition is more technical than a lemma); or a *corollary* (if it is a reasonably direct consequence of some more general theorem).

One slight variation from the typical direct line of reasoning used in most proofs (as described above) is a *proof by contradiction.* Here one wants to prove a statement P. One starts by assuming that P is false, and finding (by a direct line of reasoning as described above) that this leads to a contradiction, i.e. an absurd statement such as '$0 = 1$', or 'black is white'. One concludes that this absurdity must be the result of assuming that P is false, so one concludes that P must rather be true.

Write Sentences

Mathematical writing, particularly proofs, are written in good English (or French or German or Italian etc.). Although technical symbols may appear in a proof (and usually do), these are not central to the proof. Every proof is essentially a logical argument in defense of some statement which the author claims is true. Proper sentences are required. Remember that sentences begins with a capital letter and end with a period. Incorrect spelling, grammar and punctuation mean the proof is not even *syntactically* correct. Beyond this, the proof must also be *semantically* valid, i.e. it should be logically constructed as described above.

Symbols such as the three-dot symbol for 'therefore' (you may know the one I mean!—I can't find it in my typesetting software) may appear in some introductory books, but are no substitute for the word 'therefore' (or 'hence' or 'thus' or 'so'). This is because words have a richness and readability that symbols cannot compete with. The reason that mathematical texts, research papers, etc. use words in place of logical symbols, is not that they are written by English majors; rather, like lawyers and historians and psychologists and most others, experience shows that sentences work best.

Of course I'm not saying that the equation '$x = 1$' should be rewritten as 'the unknown quantity has the value one'. Equations such as '$x = 1$' certainly appear in symbolic form in proofs. But 'therefore' is a logical word, not an equation. A proof, as a logical argument, is expressed in English.

There are exceptions when logical symbols are used to describe statements in mathematical logic; but when they are used, they play only in a syntactic (rather than semantic) role.