



The Hyperbolic Plane

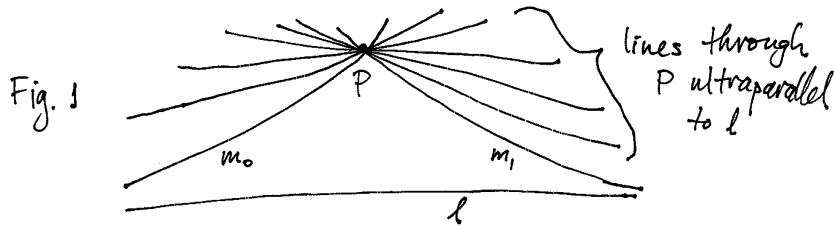
The Euclidean plane and the hyperbolic plane have many of the same notions, including distance, angle, continuity, etc. Both satisfy many of the same properties, including the following:

- Given any two distinct points P and Q , there is exactly one line passing through both P and Q .
- Given any two distinct points P and Q , the set of all points having the same distance from both P and Q is a line.
- Every line ℓ divides the plane into two connected components. Let P and Q be two points not on ℓ . We say that P and Q are *on opposite sides of ℓ* or *on the same side of ℓ* , according as the line segment PQ does or does not meet ℓ . The relation of two points being on the same side of ℓ is an equivalence relation, with two equivalence classes.
- Similarly, every point P on a line ℓ divides the other points of ℓ into two classes: those on one side of P , and those on the other side of P .
- Given any point P on a line ℓ , and any positive real number $d > 0$, there are exactly two points on ℓ at distance d from P , one on each side of P .
- If two triangles have corresponding sides of the same length, then the two triangles are congruent and there exists an isometry of the plane mapping one triangle to the other (and preserving the correspondence of sides).

However, the following axiom (a modern version of Euclid's Fifth Postulate, known as Playfair's Axiom) is satisfied by the Euclidean plane and not the hyperbolic plane:

- Given a point P not on a line ℓ , there is exactly one line m disjoint from ℓ and passing through P .

In the hyperbolic plane, there are infinitely many such lines m , two of which (pictured as m_0 and m_1 below) are 'parallel' (asymptotic) to ℓ , and a whole range of intermediate lines which are 'ultraparallel' to ℓ (each of which has positive minimum distance from ℓ).

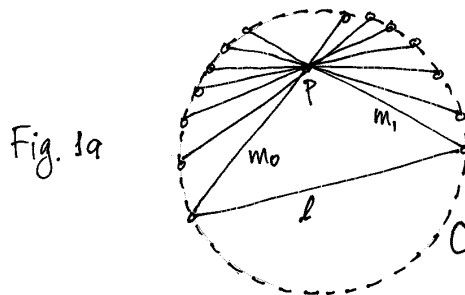


Here are some other properties of the hyperbolic plane which distinguish it from the Euclidean plane:

- The area of a triangle is less than π . The sum of its interior angles is also less than π . In fact, for a triangle with angles α , β and γ , the area is exactly the ‘angular defect’ $\pi - (\alpha + \beta + \gamma)$.
- Two lines are either intersecting, or parallel (and so they are asymptotic to each other), or they are ultraparallel (in which case they have a unique common perpendicular, joining the unique closest point of each line to the other line).
- A circle of radius r has circumference greater than $2\pi r$ (but close to $2\pi r$ if r is small). It encloses an area exceeding πr^2 (but close to πr^2 if r is small).

The Klein (or Beltrami-Klein) Model

We may represent the points of the hyperbolic plane H as points of an open disk in the Euclidean plane. Lines of H are represented as chords of the disk. The Klein picture representing Figure 1 is:

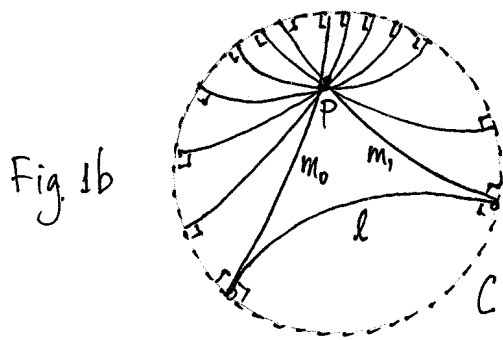


This model for H is simple but is non-conformal, i.e. it does not represent angles in H faithfully. (Of course it also does not represent distances faithfully.)

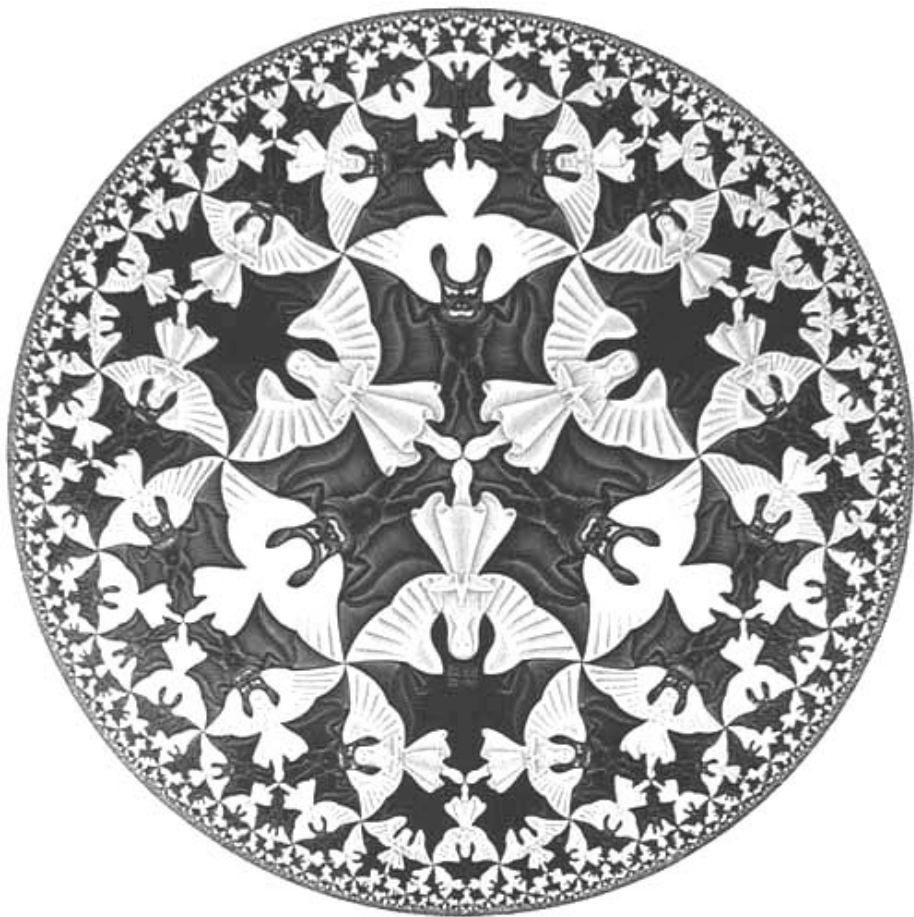
The Poincaré Disk Model

In this model for the hyperbolic plane H , points are represented again using an open disk in the Euclidean plane (the interior of a circle C). This time, lines of H are represented by circular arcs interior to C , and orthogonal to C . (We also include diameters of C , which

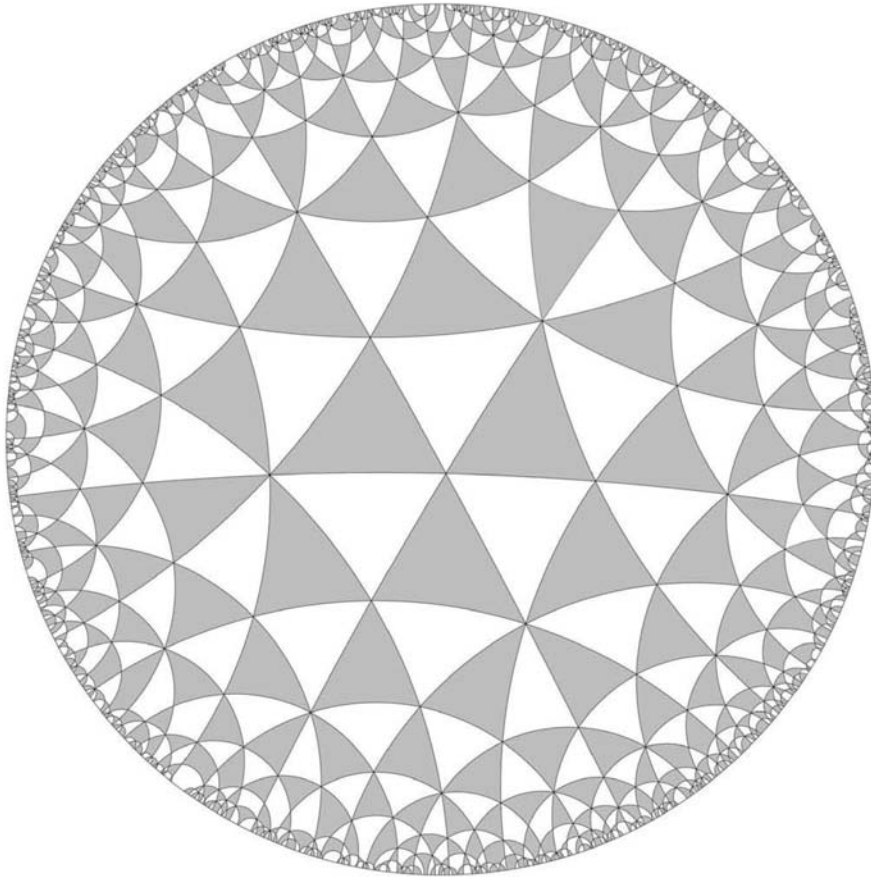
are actually limiting cases of circular arcs.) Here is the Poincaré disk representation of Fig 1:



This model distorts distances in H , but correctly represents angles: it is a *conformal* model of the hyperbolic plane. Here, in the Poincaré disk model, is pictured a tiling of the hyperbolic plane with infinitely many equal-sized angels, and equal-sized demons:

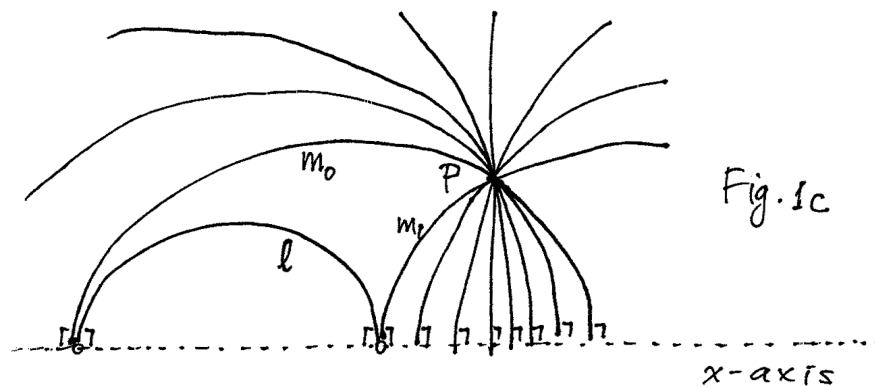


Here, in the Poincaré disk model, is pictured a tiling of the hyperbolic plane with infinitely many congruent triangles, each having angles 45° , 60° , 60° :



The Poincaré Half-Plane Model

In this model for H , we take as points the upper half-plane in \mathbb{R}^2 . Lines of H are represented by upper semicircles orthogonal to the x -axis; also (as limiting cases) upper half-lines orthogonal to the x -axis. Here is how Figure 1 is pictured in this setting:



Here, in the Poincaré half-plane model, are pictured infinitely many creatures (salamanders?) of equal size, all with perfectly straight spines, in the hyperbolic plane:



This time we go so far as to explicitly coordinatize H using the upper half-plane $U = \mathbb{R} \times (0, \infty) \subset \mathbb{R}^2$ in the language of 2-manifold. The parameterization $f : U \rightarrow H$ is global (rather than just local) and so each point of H may be coordinatized as $(x, y) = (x^1, x^2) = (u^1, u^2) \in U$. The corresponding metric tensor is chosen to be

$$g = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \frac{1}{y^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that g_p defines a positive definite symmetric matrix at each point $p = (x, y) \in H$; also the entries of g_p are smooth functions of the coordinates (x, y) as required of a Riemannian metric. Since g is a scalar multiple of the identity matrix, the representation is *conformal*, i.e. angles (although not distances) in the hyperbolic plane H are faithfully represented in the coordinate chart $U \subset \mathbb{R}^2$. We compute the inverse matrix

$$g^{-1} = \begin{bmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{bmatrix} = y^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Using the formula

$$\Gamma_{ijk} = \frac{1}{2} \left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right),$$

we obtain

$$\Gamma_{111} = \Gamma_{221} = \Gamma_{122} = \Gamma_{212} = 0; \quad \Gamma_{121} = \Gamma_{211} = \Gamma_{222} = -\frac{1}{y^3}; \quad \Gamma_{112} = \frac{1}{y^3}.$$

We raise the last index using $\Gamma_{ij}^k = g^{km} \Gamma_{ijm}$ to obtain the Christoffel symbols

$$\Gamma_{11}^1 = \Gamma_{22}^1 = \Gamma_{12}^2 = \Gamma_{21}^2 = 0; \quad \Gamma_{12}^1 = \Gamma_{21}^1 = \Gamma_{22}^2 = -\frac{1}{y}; \quad \Gamma_{11}^2 = \frac{1}{y}.$$

The geodesic equations $\ddot{u}^k + \Gamma_{ij}^k \dot{u}^i \dot{u}^j = 0$ become

$$\begin{cases} \ddot{x} - \frac{2}{y} \dot{x} \dot{y} = 0, \\ \ddot{y} + \frac{1}{y} (\dot{x})^2 - \frac{1}{y} (\dot{y})^2 = 0; \end{cases}$$

i.e.

$$\begin{cases} y\ddot{x} - 2\dot{x}\dot{y} = 0, \\ y\ddot{y} + (\dot{x})^2 - (\dot{y})^2 = 0. \end{cases}$$

To solve these equations, and thereby determine the form of geodesics in H , we consider two cases. If $\dot{x} = 0$ then we have simply

$$y\ddot{y} - (\dot{y})^2 = 0$$

and so

$$\frac{d}{dt} \left(\frac{\dot{y}}{y} \right) = \frac{y\ddot{y} - (\dot{y})^2}{y^2} = 0$$

so that

$$\frac{\dot{y}}{y} = a$$

is a real constant. Integrating once more with respect to t (and using the fact that $y > 0$) gives $y = e^{a(t-t_0)}$ for some real constant t_0 . If the parameter t represents arc length then the constant a is restricted by the condition

$$1 = g(\dot{u}, \dot{u}) = g_{ij} \dot{u}^i \dot{u}^j = \frac{1}{y^2} (\dot{x})^2 + \frac{1}{y^2} (\dot{y})^2 = 0 + a^2$$

so $a = \pm 1$. This gives a geodesic path

$$t \mapsto (x_0, e^{\pm(t-t_0)})$$

which corresponds to a vertical line in U , traversed either upwards or downwards according to the choice of sign \pm . From this we deduce that the distance between two points (x_0, y_0) and (x_0, y_1) in the hyperbolic plane is $|\ln(y_1/y_0)|$.

Next assume that $\dot{x} \neq 0$ and note that

$$\begin{aligned} \frac{d}{dt} \left(\frac{y\dot{y}}{\dot{x}} \right) &= \frac{\dot{x}y\ddot{y} + \dot{x}(\dot{y})^2 - \ddot{x}y\dot{y}}{(\dot{x})^2} \\ &= \frac{\dot{x}[y\ddot{y} + (\dot{x})^2 - (\dot{y})^2] - \dot{y}[y\ddot{x} - 2\dot{x}\dot{y}] - (\dot{x})^3}{(\dot{x})^2} \\ &= \frac{0 - 0 - (\dot{x})^3}{(\dot{x})^2} \\ &= -\dot{x}, \end{aligned}$$

i.e.

$$\frac{d}{dt} \left(\frac{y\dot{y}}{\dot{x}} + x \right) = 0.$$

Thus

$$\frac{y\dot{y}}{\dot{x}} + x = a$$

is a real constant, i.e.

$$x\dot{x} + y\dot{y} = a\dot{x}.$$

Integrating once more with respect to t gives

$$\frac{1}{2}(x^2 + y^2) = ax + b$$

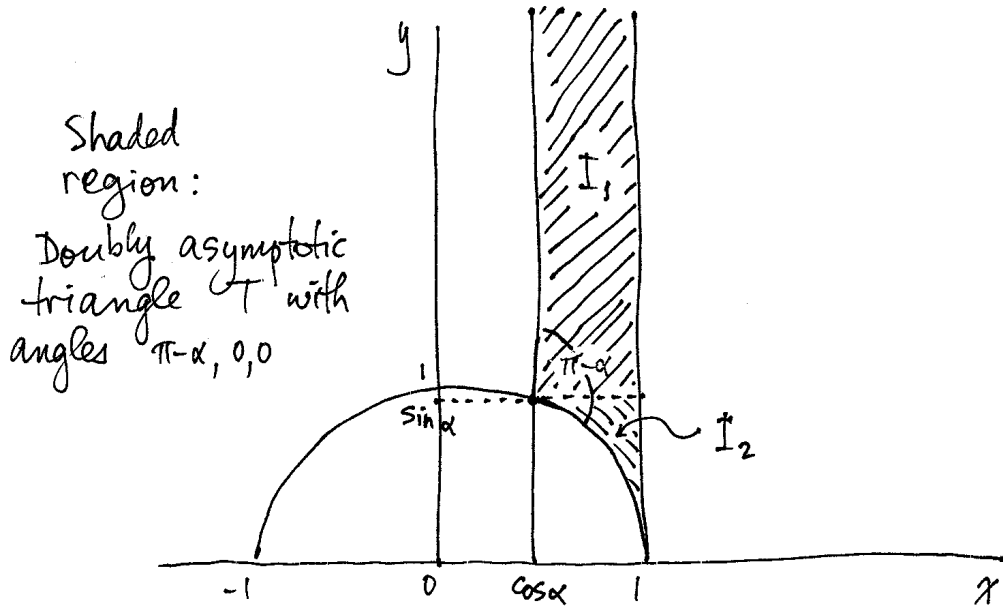
where b is another real constant; thus

$$(x - a)^2 + y^2 = a^2 + 2b.$$

In order for the geodesic path to be non-empty, we require $a^2 + 2b = r^2 > 0$, and so geodesics in H correspond to upper semicircles in U centered on the x -axis. Again one can determine the parameterization using the condition $g(\dot{u}, \dot{u}) = 1$.

Note that geodesics in H (of both types) are what we have already called the *lines* of H .

We proceed to verify that a triangle with angles α, β, γ has area equal to $\pi - (\alpha + \beta + \gamma)$. As explained in class, it suffices to consider the limiting case where two angles are zero, and the corresponding two vertices are at infinity; this is the case of a ‘doubly asymptotic triangle’. Moreover as explained in class, we can easily reduce to the case where the one remaining angle is obtuse. Now we actually consider a doubly asymptotic triangle T in H with interior angles $\pi - \alpha, 0, 0$ where $0 < \alpha \leq \pi$. We show that the area of T is the angular defect α as predicted. After applying a suitable isometry if necessary, we may assume that the triangle T represented in the half-plane model as shown in the Poincaré half-plane model:



The area of T is

$$\iint_T \sqrt{\det(g)} \, dx \, dy = I_1 + I_2$$

where the integrals I_1 and I_2 are found by decomposing T into the two regions shown above. We evaluate

$$I_1 = \int_{\sin \alpha}^{\infty} \int_{\cos \alpha}^1 \frac{1}{y^2} \, dx \, dy = \int_{\sin \alpha}^{\infty} \frac{1 - \cos \alpha}{y^2} \, dy = \frac{1 - \cos \alpha}{\sin \alpha};$$

$$\begin{aligned} I_2 &= \int_0^{\sin \alpha} \int_{\sqrt{1-y^2}}^1 \frac{1}{y^2} \, dx \, dy = \int_0^{\sin \alpha} \frac{1 - \sqrt{1-y^2}}{y^2} \, dy = \int_0^{\sin \alpha} \frac{dy}{1 + \sqrt{1-y^2}} \\ &= \left[\sin^{-1} y - \frac{y}{1 + \sqrt{1-y^2}} \right]_0^{\sin \alpha} = \alpha - \frac{\sin \alpha}{1 + \cos \alpha} = \alpha - \frac{1 - \cos \alpha}{\sin \alpha}. \end{aligned}$$

Consider a fractional linear transformation $\varphi : H \rightarrow H$ of the form

$$\varphi(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{R}$ with $ad - bc > 0$, and we express points of H as complex numbers $z = x + iy$ with $y > 0$. We verify that $\varphi : H \rightarrow H$ is an isometry of the hyperbolic plane.

Note that

$$\varphi(x, y) = (u, v) = \left(\frac{ac(x^2 + y^2) + (ad + bc)x + bd}{c^2(x^2 + y^2) + 2cdx + d^2}, \frac{(ad - bc)y}{c^2(x^2 + y^2) + 2cdx + d^2} \right).$$

The Jacobian of φ at $p = (x, y) \in H$ is

$$Df|_p = \frac{ad - bc}{(c^2(x^2 + y^2) + 2cdx + d^2)^2} \begin{bmatrix} (cx + d)^2 - c^2y^2 & 2(cx + d)cy \\ -2(cx + d)cy & (cx + d)^2 - c^2y^2 \end{bmatrix}.$$

The first fundamental form at $p = (x, y) \in H$ is

$$g_p(X, Y) = \frac{1}{y^2} [\xi^1 \quad \xi^2] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \eta^1 \\ \eta^2 \end{bmatrix}$$

where $X = \xi^i \frac{\partial}{\partial x^i}$ and $Y = \eta^i \frac{\partial}{\partial x^i}$. Using the first fundamental form at $\varphi(p) = (u, v) \in H$ we see that

$$\begin{aligned} g_{\varphi(p)}(Df|_p(X), Df|_p(Y)) &= \frac{1}{v^2} [\xi^1 \quad \xi^2] (Df|_p)^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (Df|_p) \begin{bmatrix} \eta^1 \\ \eta^2 \end{bmatrix} \\ &= \frac{1}{v^2} [\xi^1 \quad \xi^2] \begin{bmatrix} \frac{(ad-bc)^2}{(c^2(x^2+y^2)+2cdx+d^2)^2} & 0 \\ 0 & \frac{(ad-bc)^2}{(c^2(x^2+y^2)+2cdx+d^2)^2} \end{bmatrix} \begin{bmatrix} \eta^1 \\ \eta^2 \end{bmatrix} \\ &= \frac{1}{y^2} [\xi^1 \quad \xi^2] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \eta^1 \\ \eta^2 \end{bmatrix} \\ &= g_p(X, Y). \end{aligned}$$

(Thanks to MAPLE!) Thus $\varphi : H \rightarrow H$ is an isometry.

Another fractional linear transformation can be used to show that the Poincaré half-plane model is isometric to the Poincaré disk model (if a suitable metric is chosen on the disk).