University of Wyoming
Math 3700—Spring 2016
Combinatorics
Department of Mathematics

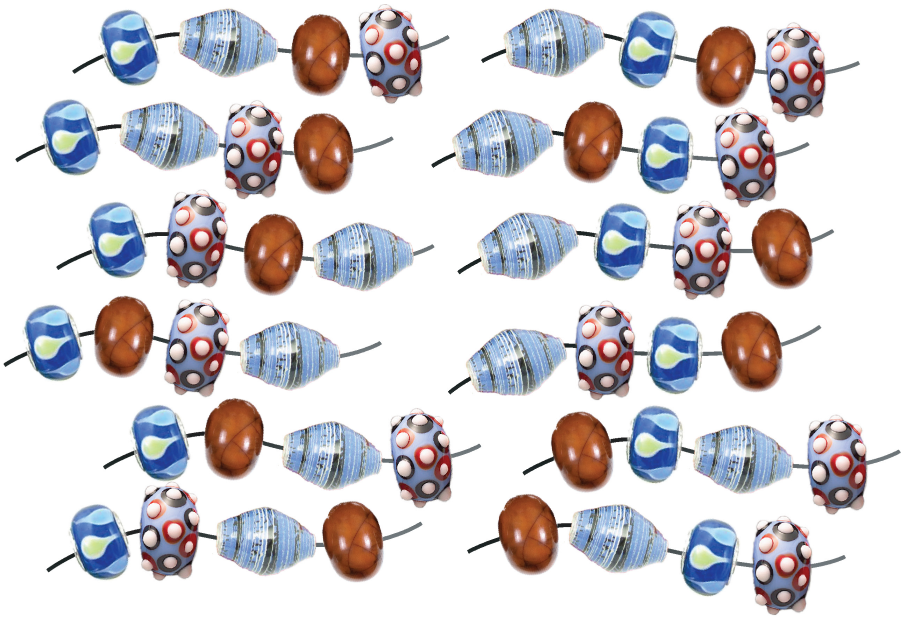
Counting Necklaces (Handout March, 2016)

We illustrate the use of exponential generating functions in a counting problem. More details are found in Chapter 8 of the textbook, although we will be covering only a few highlights of this topic.

How many ways can we string together n different beads to form a necklace? Start by stringing $n \geqslant 2$ beads onto a linear piece of string (i.e. a cord with two end points). There are $n!$ choices of order for this sequence; but the reverse sequence gives exactly the same string of beads, so really there are

$$\begin{cases} \frac{n!}{2} & \text{if } n \geqslant 2; \\ 1, & \text{if } n = 1 \end{cases}$$

ways to string n beads onto a linear piece of string. (The case $n = 1$ is an exception, with only one way to string one bead, since reversing a string of one bead gives the same sequence.) Here we picture the $\frac{4!}{2} = 12$ ways to string 4 beads in a row:



Now join the ends ot the string together to form a loop. If $n \geqslant 3$, there are n different linear strings that could be used to form the same loop necklace (in other words, for each loop necklace there are n different points to break the string to form a linear string with n

beads). The cases $n \leqslant 2$ are again exceptional. So the number of loop necklaces that can be formed using n different beads is

$$a_n = \begin{cases} \frac{n!}{2n} = \frac{(n-1)!}{2} & \text{if } n \geqslant 3; \\ 1, & \text{if } n = 1, 2. \end{cases}$$

By convention, we will take $a_0 = 0$ (with no beads, we cannot form a necklace at all). Here are the $\frac{(4-1)!}{2} = 3$ loop necklaces using 4 beads:

The exponential generating function for a_n is

$$A(x) = \sum_{n\geqslant 1} \frac{a_n}{n!}x^n = x + \tfrac{1}{2}x^2 + \sum_{n\geqslant 3} \frac{(n-1)!x}{2n!} = x + \tfrac{1}{2}x^2 + \tfrac{1}{2}\Big(\frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \cdots\Big).$$

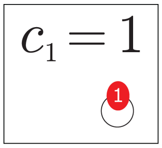
To find a closed formula for $A(x)$, consider the derivative

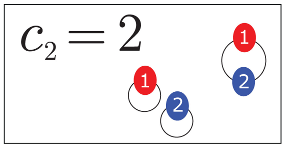
$$\begin{aligned} A'(x) &= 1 + x + \tfrac{1}{2}\big(x^2 + x^3 + x^4 + x^5 + \cdots\big) = \tfrac{1}{2} + \tfrac{1}{2}x + \tfrac{1}{2}\big(1 + x + x^2 + x^3 + \cdots\big) \\ &= \tfrac{1}{2} + \tfrac{1}{2}x + \tfrac{1}{2}\Big(\frac{1}{1-x}\Big). \end{aligned}$$

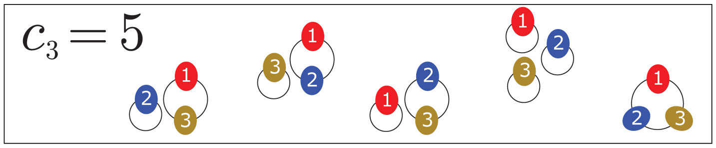
Integrating, we obtain

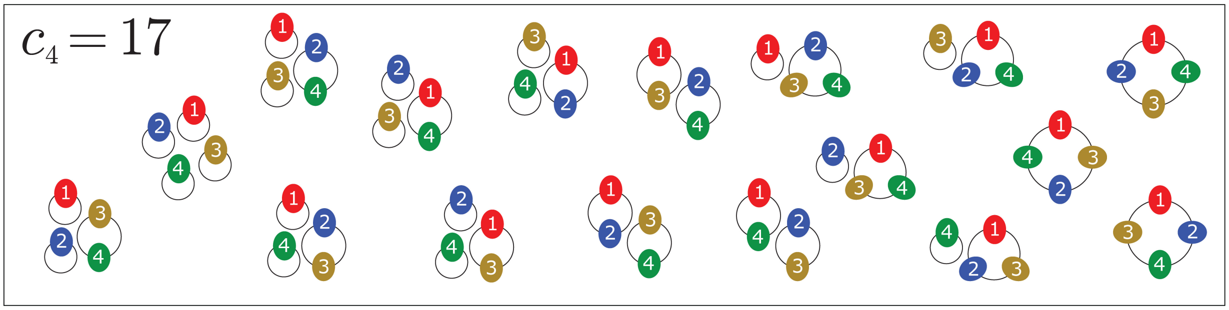
$$A(x) = \frac{x}{2} + \frac{x^2}{4} - \tfrac{1}{2}\ln(1-x).$$

Now let c_n be the number of ways to form a collection of nonempty necklaces from n different beads. Call this number c_n. After $c_0 = 1$ the next few terms are as shown:









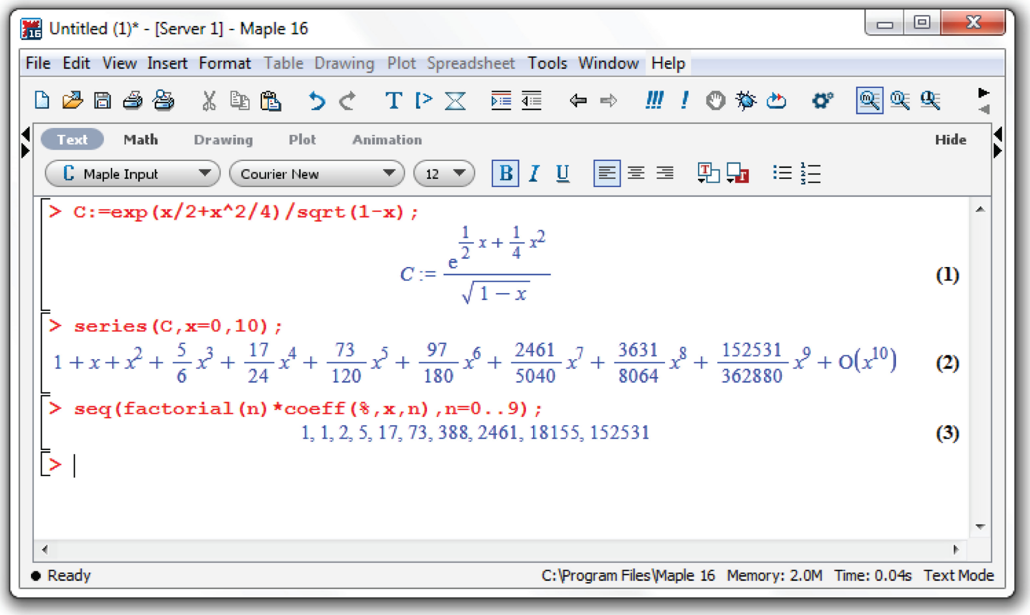
No additional structure is required on the set of necklaces, so we take $b_n = 1$ and

$$B(x) = \sum_{n\geqslant 0} \frac{b_n}{n!} x^n = \sum_{n\geqslant 0} \frac{x^n}{n!} = e^x.$$

Finally, the exponential generating function of c_n is

$$C(x) = B(A(x)) = \frac{e^{\frac{x}{2}+\frac{x^2}{4}}}{\sqrt{1-x}} = 1 + x + x^2 + \frac{5}{6}x^3 + \frac{17}{24}x^4 + \frac{73}{120}x^5 + \frac{97}{180}x^6 + \cdots$$

where we are able to determine as many terms as desired in this series expansion using the MAPLE session



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> C:=exp(x/2+x^2/4)/sqrt(1-x);
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$$C := \frac{e^{\frac{1}{2}x+\frac{1}{4}x^2}}{\sqrt{1-x}} \qquad (1)$$

```
> series(C,x=0,10);
```

$$1 + x + x^2 + \frac{5}{6}x^3 + \frac{17}{24}x^4 + \frac{73}{120}x^5 + \frac{97}{180}x^6 + \frac{2461}{5040}x^7 + \frac{3631}{8064}x^8 + \frac{152531}{362880}x^9 + O(x^{10}) \qquad (2)$$

```
> seq(factorial(n)*coeff(%,x,n),n=0..9);
```

$$1, 1, 2, 5, 17, 73, 388, 2461, 18155, 152531 \qquad (3)$$

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> |
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From this we are able to recover the value of c_n for small values of n by multiplying the coefficient of x^n in the series expansion, by $n!$ as we have demonstrated in our MAPLE session, thereby obtaining

n	0	1	2	3	4	5	6	7	8	9
c_n	1	1	2	5	17	73	388	2461	18155	152531

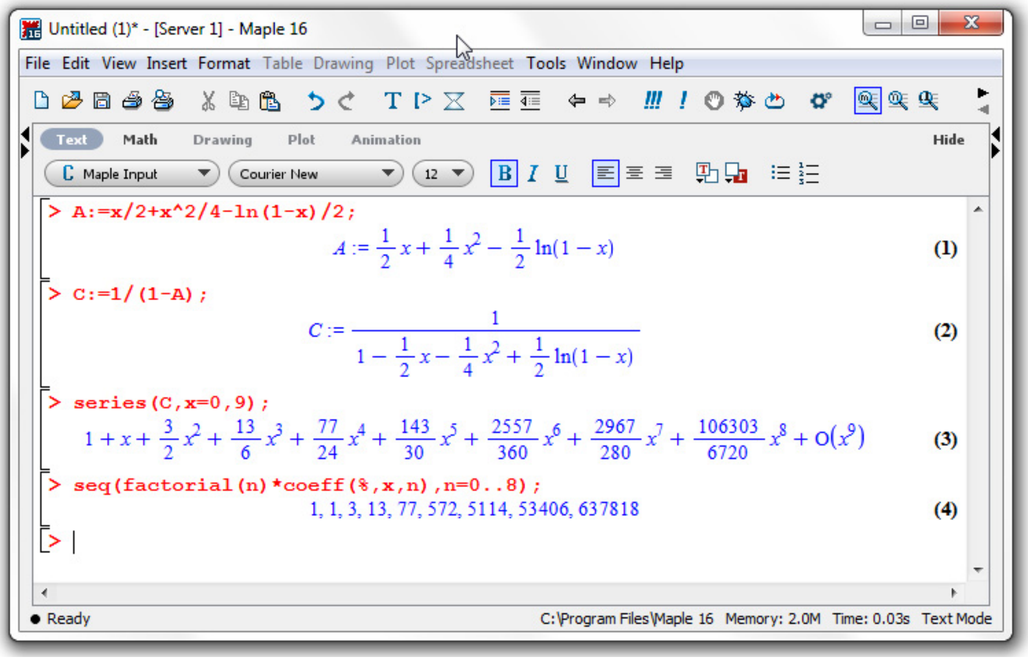
A second example, in which we incorporate more structure on the set of necklaces, is the following: Denote by c_n the number of ways to construct a set of loop necklaces from n different beads, and then arrange these necklaces in a row in order. We compute the first few terms in this new sequence by a slight modification of our first example. There are now $b_n = n!$ ways to list n objects in a row, and the exponential generating function for this is

$$B(x) = \sum_{n\geqslant 0} \frac{b_n}{n!} x^n = \sum_{n\geqslant 0} x^n = \frac{1}{1-x}\,.$$

Using the same $A(x)$ as above, we now obtain

$$C(x) = B(A(x)) = \frac{1}{1 - \frac{x}{2} - \frac{x^2}{4} + \frac{1}{2}\ln(1-x)} = 1 + x + \frac{3}{2}x^2 + \frac{13}{6}x^3 + \frac{77}{24}x^4 + \cdots$$

Here is our MAPLE session:



```
> A:=x/2+x^2/4-ln(1-x)/2;
```

$$A := \frac{1}{2}x + \frac{1}{4}x^2 - \frac{1}{2}\ln(1-x) \tag{1}$$

```
> C:=1/(1-A);
```

$$C := \frac{1}{1 - \frac{1}{2}x - \frac{1}{4}x^2 + \frac{1}{2}\ln(1-x)} \tag{2}$$

```
> series(C,x=0,9);
```

$$1 + x + \frac{3}{2}x^2 + \frac{13}{6}x^3 + \frac{77}{24}x^4 + \frac{143}{30}x^5 + \frac{2557}{360}x^6 + \frac{2967}{280}x^7 + \frac{106303}{6720}x^8 + \mathrm{O}(x^9) \tag{3}$$

```
> seq(factorial(n)*coeff(%,x,n),n=0..8);
```

$$1, 1, 3, 13, 77, 572, 5114, 53406, 637818 \tag{4}$$

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> |
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As before, the first few values of c_n are found to be

n	0	1	2	3	4	5	6	7	8
c_n	1	1	3	13	77	572	5114	53406	637818

We verify the smallest values here using our previous catalog of necklaces:

$$\begin{aligned}
c_0 &= 1;\\
c_1 &= 1;\\
c_2 &= 1\times 2! + 1\times 1! = 2 + 1 = 3;\\
c_3 &= 1\times 3! + 3\times 2! + 1\times 1! = 6 + 6 + 1 = 13;\\
c_4 &= 1\times 4! + 6\times 3! + 7\times 2! + 3\times 1! = 24 + 36 + 14 + 3 = 77.
\end{aligned}$$

Let us explain c_3: We have listed five ways to construct necklaces from 3 different beads,

- one way using three necklaces, which can be listed in $3! = 6$ ways;
- three ways using two necklaces, in each case with $2!$ ways to list the necklaces, for a total of $3\times 2! = 6$ ways; and
- one way using a single necklace, which can be listed in only $1!$ way,

for a total of $c_3 = 6 + 6 + 1 = 13$ ways to construct necklaces from 3 different beads, then list them in some order.