The Alon-Tarsi Conjecture

G. Eric Moorhouse

Department of Mathematics
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5 Dec 2008 / RMAC Seminar

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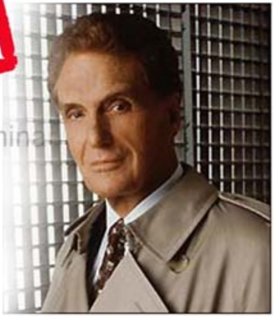

Latin Squares

A *Latin square of order n* is an $n \times n$ array in which each of the symbols $1, 2, \ldots, n$ occurs once in each row and in each column. Denote $LS(n) = \{$Latin squares of order $n\}$.

$L \in LS(n)$ is *row-even* (or *row-odd*) according as

$$\mathrm{sgn}(\sigma_1\sigma_2\cdots\sigma_n) = +1 \text{ or } -1,$$

resp., where $\sigma_1, \sigma_2, \ldots, \sigma_n \in S_n$ are the rows of L.
Similarly *column-even*, *column-odd*.
L is *even* or *odd* according to its *sign*:

$$\mathrm{sgn}(L) = \mathrm{sgn}(\sigma_1\sigma_2\cdots\sigma_n\tau_1\tau_2\cdots\tau_n)$$

where $\sigma_i \in S_n$ are the columns, and $\tau_j \in S_n$ are the rows, of L.

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$$L = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$$

is row-even,

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Alon-Tarsi Conjecture

n	$\lvert ELS(n)\rvert$	$\lvert OLS(n)\rvert$
1	1	0
2	2	0
3	6	6
4	576	0
5	80640	80640
6	505958400	306892800
7	30739709952000	30739709952000
8	55019078005712486400	53756954453370470400

Conjecture (Alon and Tarsi, 1986)

For even $n \geqslant 2$, we have $|ELS(n)| \neq |OLS(n)|$.

Equivalently, there are unequal numbers of row-even and row-odd Latin squares.

N.B. There is also an apparently unrelated *Alon-Tarsi Basis Conjecture*.

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Extended Alon-Tarsi Conjecture

$FDLS(n) = \{\text{Latin squares of order } n \text{ with 1's on main diagonal}\}$

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$FDOLS(n) = \{\textit{odd} \text{ Latin squares of order } n \text{ with 1's on main diagonal}\}$

$$AT(n) = \frac{|FDELS(n)| - |FDOLS(n)|}{(n-1)!}$$

n	1	2	3	4	5	6	7
$AT(n)$	1	-1	4	-24	2304	368640	6210846720

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$$|ELS(n)| - |OLS(n)| = \begin{cases} n!(n-1)!AT(n), & \text{for even } n > 1; \\ 0, & \text{for odd } n > 1; \\ 1, & \text{for } n = 1. \end{cases}$$

Extended Alon-Tarsi Conjecture (Zappa, 1997)

For every $n \geqslant 1$, we have $AT(n) \neq 0$.

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Rota's Basis Conjecture

Conjecture (Rota, 1992)

Let

$$\begin{array}{l} \mathcal{B}_1 = \{v_{11}, \quad v_{12}, \quad \ldots, \quad v_{1n}\}, \\ \mathcal{B}_2 = \{v_{21}, \quad v_{22}, \quad \ldots, \quad v_{2n}\}, \\ \qquad\quad \vdots \qquad\quad \vdots \qquad\quad \ddots \qquad \vdots \\ \mathcal{B}_n = \{v_{n1}, \quad v_{n2}, \quad \ldots, \quad v_{nn}\} \end{array}$$

be bases for an n-dimensional vector space V. Then the vectors in each row can be permuted in such a way that the columns also form a set of n bases for V.

Rota has also conjectured a generalization to matroids of rank n, which has been verified only for $n = 3$.

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Theorem (Huang and Rota (1992))

Let $n \geq 2$ be even, and let F be a field of characteristic zero. If $|ELS(n)| \neq |OLS(n)|$, then Rota's conjecture holds for F^n.

Proof uses bracket algebras.

Theorem (Onn (1997))

Let $n \geq 2$ be even, and let F be a field whose characteristic does not divide $|ELS(n)| - |OLS(n)|$. Then Rota's conjecture holds for F^n.

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Conjecture (Kahn, 1991)

Let V be an n-dimensional vector space, and consider an $n \times n$ array of bases $\mathcal{B}_{ij}$ for V. Then we may choose $v_{ij} \in \mathcal{B}_{ij}$ such that the $n \times n$ array of vectors v_{ij} has a basis for V in every row and column.

This generalizes both the Alon-Tarsi Conjecture and Rota's Basis Conjecture.

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Dinitz' Problem

Conjecture (Dinitz, 1979)

Given an $n \times n$ array of sets S_{ij} of cardinality $|S_{ij}| = n$, there exists a *partial Latin square* L of order n (i.e. no repeated symbols in any row or column) such that $L_{ij} \in S_{ij}$.

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Proved by Galvin (1995) using results of Janssen (1993).

Janssen had solved the problem affirmatively for $(n-1) \times n$ partial Latin rectangles, using work of Alon and Tarsi.

Alon and Tarsi (1986) had showed that their conjecture for even n, implies Dinitz' Conjecture.

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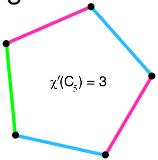
List Colouring Conjecture

The *edge chromatic number* $\chi'(\Gamma)$ of a graph Γ is the minimum number of colours in a proper colouring of the edges of Γ (i.e. no two edges of the same colour meeting at a vertex).



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Clearly $ch'(\Gamma) \geqslant \chi'(\Gamma)$.

Conjecture (predates 1995; attribution unclear)

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The Dinitz Conjecture regards the special case of a complete bipartite graph: $ch'(K_{n,n}) = n = \chi(K_{n,n})$.

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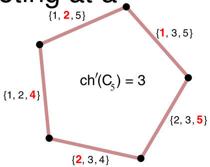

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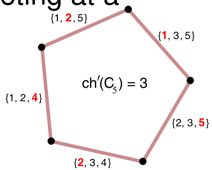

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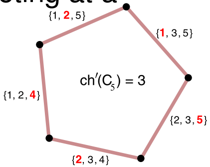

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Tensors ("Hypermatrices")

An *r-tensor of dimension n* (and *rank r*) is an $\overbrace{n \times n \times \cdots \times n}^{r}$ array of scalars

$$M = \left(m_{i_1, i_2, \ldots, i_r}\right), \qquad 1 \leqslant i_1, i_2, \ldots, i_r \leqslant n.$$

Its *Cayley "hyperdeterminant"* is

$$\det(M) = \sum_{\sigma_2, \sigma_3, \ldots, \sigma_r \in S_n} \operatorname{sgn}(\sigma_2\sigma_3\cdots\sigma_r) m_{1,\sigma_2(1),\sigma_3(1),\cdots,\sigma_2(1)}$$

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Levi-Civita Tensor ("Determinant tensor")

$$\mathcal{E}_{i_1,i_2,\ldots,i_n} = \begin{cases} 0, & \text{if two of } i_1, i_2, \ldots, i_n \text{ coincide;} \\ \operatorname{sgn}(\sigma), & \text{if } (i_1, i_2, \ldots, i_n) \text{ is a permutation } \sigma \text{ of } (1, 2, \ldots, n). \end{cases}$$

$$n! \det \mathcal{E} = |ELS(n)| - |OLS(n)|$$

Theorem (Zappa, 1997)

(i) *If* $n \geqslant 2$ *is even, then* $AT(n) \geqslant 0$.

(ii) *If* n *is even and* $AT(n) \neq 0$, *then* $AT(2n) \neq 0$.

(iii) *If* n *is odd and* $AT(n)AT(n+1) \neq 0$, *then* $AT(2n) \neq 0$.

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Levi-Civita Tensor (“Determinant tensor”)

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$$L \mapsto C^{-1}LC.$$

Note that the action of G on $FDLS(p)$ preserves signs, hence preserving the sum

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Even vs. odd latin squares

view revisions

A latin square is *even* if the product of the signs of all of the row and column permutations is 1 and is *odd* otherwise.

Conjecture For every positive even integer n, the number of even latin squares of order n and the number of odd latin squares of order n are different.

Importance: High ★★★
Author(s): Alon, Noga
Tarsi, Michael
Subject: Combinatorics
Keywords: latin square
Recomm. for undergrads: no
Posted by: mdevos
on: October 7th, 2007

For every positive integer n, let $ELS(n)$, $(OLS(n))$ be the number of even (odd) latin squares of order n.

The inspiration for this conjecture comes from an attempt by Alon and Tarsi to use their polynomial technique to show that the complete bipartite graph $K_{n,n}$ is n-edge-choosable (a famous conjecture of Dinitz asserts that this is always true). They show (in [AT]) that whenever $ELS(n) \neq OLS(n)$, the graph $K_{n,n}$ is n-edge-choosable. For odd integers $n > 1$ it is easy to see that $ELS(n) = OLS(n)$, since interchanging the first two rows has no effect on the signs of the rows, but flips the signs of all of the columns. For even n, Alon and Tarsi checked that $ELS(n)$ and $OLS(n)$ were different for $n = 2, 4, 6$ and conjectured that this pattern would continue. Although Dinitz' Conjecture has since been resolved, Alon and Tarsi's conjecture remains quite interesting. In particular, it has been shown by Huang and Rota [HR] that the truth of this conjecture would imply Rota's basis conjecture for even values of n (see [O] for a nice proof of this).

ELS() and OLS() appear in the The Encyclopedia of Integer Sequences as A114628 and A114629. The following chart shows the first few values. Although the data here is quite limited, $ELS(n) > OLS(n)$ for every even n in the chart, and as far as we know, this might hold in general.

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