The Maslov Index & Covers of Symplectic Dual Polar Graphs

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Let F be a field, $\operatorname{char} F \neq 2$; V a $2n$-dimensional vector space over F; $B: V \times V \to F$ a symplectic form.

A subspace $L < V$ is maximal totally isotropic if

$$L^{\perp} := \{v \in V : B(u,v) = 0 \text{ for all } u \in L\} = L.$$

Such subspaces all have dimension n and are called Lagrangians. The set $\Lambda_n(F)$ of all such subspaces is the Lagrangian Grassmannian. It is a graph ($L_1 \sim L_2$ iff $L_1 \cap L_2$ has codimension 1 in L_1, L_2) in general (with possibly more structure).

This talk is about covers of the graph $\Lambda_n(F)$ and the group $Sp_{2n}(F) \leq \operatorname{Aut} \Lambda_n(F)$.

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The Metaplectic Group and the Weil Representation

Consider the Heisenberg group H with elements $(v,a)\in V\times F$ and product

$$(v,a)(w,b) = (v+w,\ a+b+B(v,w)).$$

Denote $Z = Z(H) = H' = \{0\}\times F$. Maximal abelian subgroups of H have the form $LZ = L\times Z$, $L\in\Lambda_n(F)$. Take any nontrivial additive character χ of F, i.e.

$$\chi: F\to\mathbb{C},\quad |\chi(a)|=1,\quad \chi(a+b)=\chi(a)\chi(b),\quad \chi\neq 1.$$

Fix $L\in\Lambda_n(F)$ and extend χ trivially to a linear character of $LZ=L\times Z$ via

$$\chi(u,a)=\chi(a) \quad\text{for } (u,a)\in L\times Z.$$

Inducing χ from LZ up to H gives a representation $\eta: H\to GL(\mathbb{C}^L)$ where $\mathbb{C}^L=\{\text{maps } L\to\mathbb{C}\}$. The associated character $\operatorname{tr}\eta(v,a)$ vanishes for $v\neq 0$; on Z it is uniquely determined by χ. Now $G=Sp_{2n}(F)$ acts on H via $g(v,a)=(gv,a)$. Twisting η by g gives another representation $\eta_g(v,a)=\eta(v,a)$ with the same character as η. So

$$\eta_g(v,a) = W(g)\,\eta(v,a)\,W(g)^{-1}$$

for some $W: G\to GL(\mathbb{C}^L)$. This is a projective representation of $G=Sp_{2n}(F)$. Its image in $GL(\mathbb{C}^L)$ is the metaplectic group $Mp_{2n}(F)$.

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The central extension

$$1 \longrightarrow Z(Mp_{2n}(F)) \longrightarrow Mp_{2n}(F) \longrightarrow Sp_{2n}(F) \longrightarrow 1$$

is a nonsplit double cover if $F = \mathbb{R}$. For $F = \mathbb{F}_q$ or $\mathbb{C}$, W can be 'linearized' and $Mp_{2n}(F) = Sp_{2n}(F)$.

For $F = \mathbb{F}_q$, $W : Sp_{2n}(q) \to GL(\mathbb{C}^L)$ has degree q^n and is independent of L, χ (up to equivalence). It has two irreducible constituents

$$W(g) = W_+(g) \oplus W_-(g)$$

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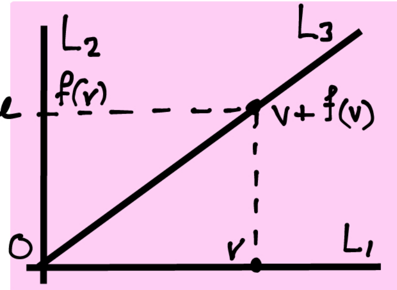
The Maslov Index

Given a triple L_1, L_2, L_3 of mutually transverse Lagrangians, i.e. $L_i \cap L_j = 0$ for $i \neq j$, there is a unique $f : L_1 \xrightarrow{\cong} L_2$ such that

$$L_3 = \{ v + f(v) : v \in L_1 \}.$$



Since L_3 is totally isotropic,

$$B(u+f(u),\, v+f(v)) = 0 \quad \text{for all } u, v \in L_1,$$

i.e. $B(u, f(v)) = B(v, f(u))$, $u, v \in L_1$. Since char $F \neq 2$, this symmetric bilinear form on L_1 gives a unique quadratic form $Q(u) = \frac{1}{2} B(u, f(u))$, $u \in L_1$. Define the Maslov index $\tau(L_1, L_2, L_3)$ to be the class of this quadratic form in $\mathrm{Witt}(F)$, the Witt ring of F.

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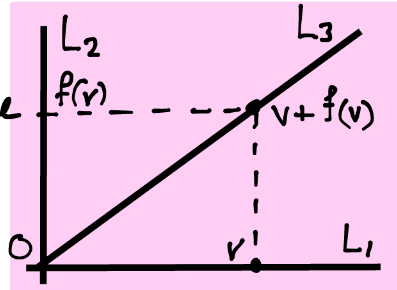

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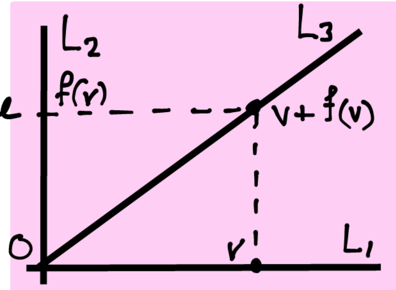

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The Maslov Index

The surprising fact is that the Maslov index $\tau(L_1, L_2, L_3)$ can be defined for all Lagrangian triples, not just for mutually transverse Lagrangians. It is an invariant:

$$\tau(gL_1, gL_2, gL_3) = \tau(L_1, L_2, L_3) \text{ for all } g \in G = Sp_{2n}(F).$$

In fact,

Theorem. Two triples (L_1, L_2, L_3) and (L_1', L_2', L_3') are in the same orbit under $G = Sp_{2n}(F)$ iff they are in the same orbit under $GL_{2n}(F)$ and $\tau(L_1, L_2, L_3) = \tau(L_1', L_2', L_3')$.

The Maslov index satisfies a cocycle condition:

Theorem: $\tau(L_1, L_2, L_3) - \tau(L_1, L_2, L_4) + \tau(L_1, L_3, L_4) - \tau(L_2, L_3, L_4) = 0.$

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The 2-cocycle on $G = Sp_{2n}(F)$

Fix an arbitrary Lagrangian and take

$$\tau(g,h) := \tau(L, gL, ghL) \quad \text{for } g,h \in G.$$

Theorem. $\tau(g,g') - \tau(gg',g'') + \tau(g,g'g'') - \tau(g',g'') = 0.$

So we get a central extension

$$1 \longrightarrow \mathrm{Witt}(F) \longrightarrow M \longrightarrow G \longrightarrow 1$$

where M consists of pairs $(g,Q) \in G \times \mathrm{Witt}(F)$ and

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The metaplectic group $Mp_{2n}(F)$ is (involved in) this group M.

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Double covers of the graph $\Lambda_n(F)$, $F=\mathbb{F}_q$, $q\equiv 1 \bmod 4$

(Williford and M., 2015) For $L, L' \in \Lambda_n(F)$ we explicitly define $\sigma(L,L') = \sigma(L',L) = \pm 1$ such that

$$\sigma(L,L',L'') := \sigma(L,L')\,\sigma(L',L'')\,\sigma(L'',L)$$

coincides with the Maslov index $\tau(L,L',L'')$ as defined above.

Vertices of the double cover $\tilde{\Lambda}_n(F) \to \Lambda_n(F)$ are pairs $(L,\varepsilon) \in \Lambda_n(F) \times \{\pm 1\}$ with adjacency

$$(L,\varepsilon)\sim(L',\varepsilon') \iff \begin{cases} L\sim L' \ \text{ i.e. } \dim L/_{L\cap L'} = \dim L'/_{L\cap L'} = 1 \\ \text{and} \\ \varepsilon\varepsilon' = \sigma(L,L'). \end{cases}$$

This is but one of the relations in a $(2n+1)$-class association scheme naturally defined on $\tilde{\Lambda}_n(F)$, invariant under $2\times G$.

This scheme arises from a two-graph on $\Lambda_n(F)$ invariant under G, namely the set of triples (L,L',L'') having $\sigma=+1$.

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The case of $\Lambda_n(F)$, $F=\mathbb{F}_q$, $q\equiv 3 \bmod 4$

$$\tau(L',L'',L) = \tau(L,L',L'') = -\tau(L',L,L'')$$

i.e. on $\Lambda_n(F)$ we have a naturally defined skew two-graph invariant under $G=Sp_{2n}(F)$. This gives rise to a $(2n+1)$-class commutative (but non-symmetric) association scheme with vertices $(L,\varepsilon)\in\Lambda_n(F)\times\{\pm1\}$ as before.

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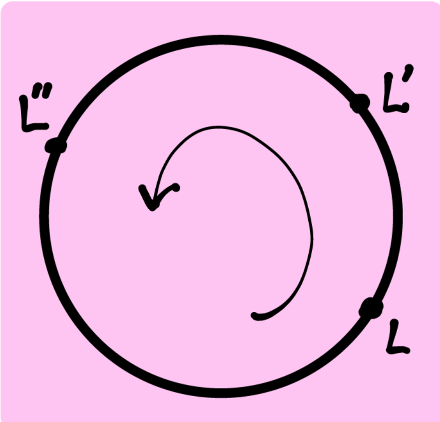
$\Lambda_1(\mathbb{R}) \simeq S^1$



$\tau(L, L', L'') = \pm 1$ according as the triple is oriented counter-clockwise or clockwise

$\Lambda_1(\mathbb{F}_q)$, $q \equiv 3 \bmod 4$

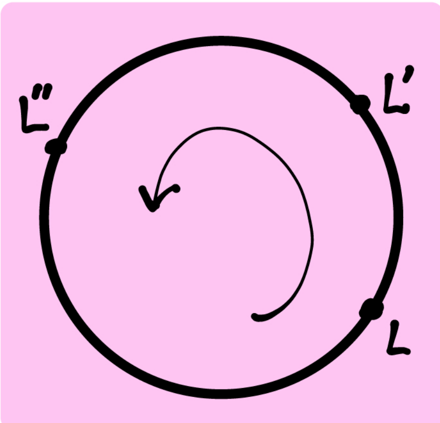
$q = 7$:

$$\tau(L_i, L_j) = \begin{cases} 1 & \text{iff } j-i = \square \\ -1 & \text{iff } j-i = \not\square \end{cases}$$

$L_i \to L_j \iff \tau(L_i, L_j) = +1$

$\tau(L, L', L'') = \tau(L, L')\,\tau(L', L'')\,\tau(L'', L')$

$\Lambda_1(\mathbb{R}) \simeq S^1$	$\Lambda_1(\mathbb{F}_q)$, $q \equiv 3 \bmod 4$



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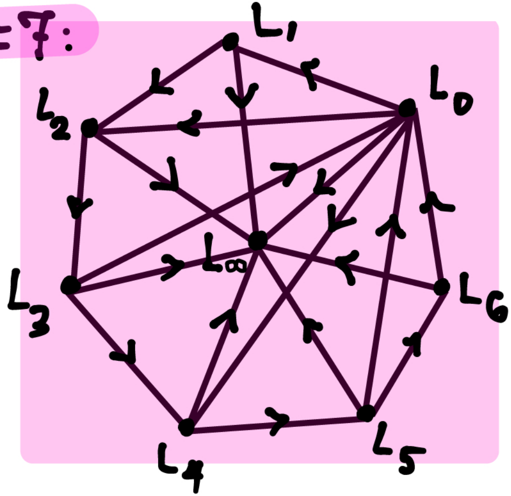

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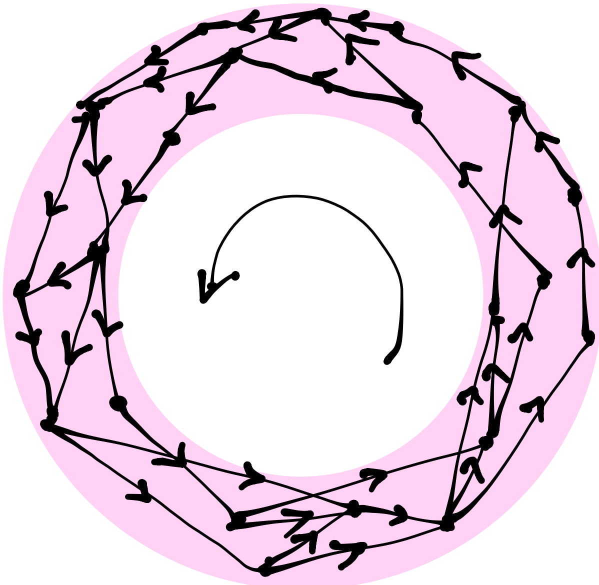
$\Lambda_n(\mathbb{R})$



$\pi_1(\Lambda_n(\mathbb{R})) \cong \mathbb{Z}$

From the 1-dimensional perspective, $\Lambda_n(\mathbb{R})$ looks like a circle.

References

V.P. Maslov (1965), Theory of perturbations and asymptotic methods.

V.I. Arnold (1967) On a characteristic class entering into conditions of quantisation.

G. Lion and M. Vergne (1980) The Weil Representation, Maslov Index and Theta Series.

A. Weil (1964), Sur certain groupes d'opérateurs unitaires.

S.E. Cappell et al. (1994), On the Maslov index.

P. Gérardin (1977) Weil representations associated to finite fields.

P. Piccione and D.V. Tausk (2008), A Student's Guide to Symplectic Spaces, Grassmannians and Maslov Index.

J. Robbin and D. Salamon (1993), The Maslov index for paths.

T. Thomas (2008), The character of the Weil representation.