Double Covers

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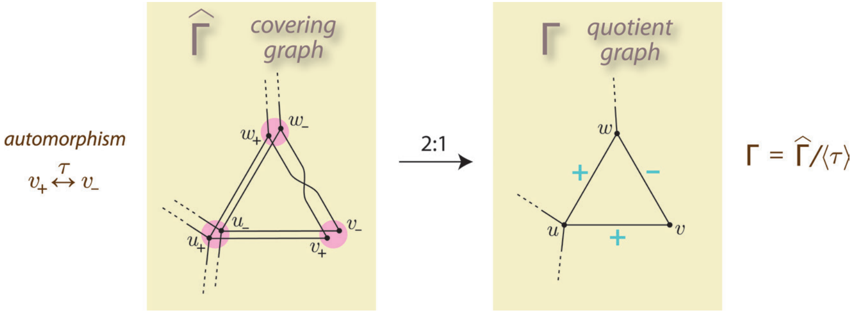
26 November 2017

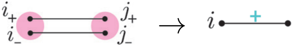
Double Covers of Graphs

A double cover is a surjective graph homomorphism $\theta : \widehat{\Gamma} \to \Gamma$ which is 2-to-1 both on vertices and on edges.



Each vertex $v \in \Gamma$ lifts to a fibre $\theta^{-1}(v) = \{v_+, v_-\}$ of size two in $\widehat{\Gamma}$. Each edge $\{v, w\}$ in Γ lifts to a matching between the fibres $\{v_+, v_-\}$ and $\{w_+, w_-\}$.

Two special cases

If *every* edge is covered as 

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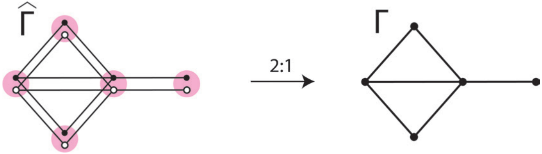

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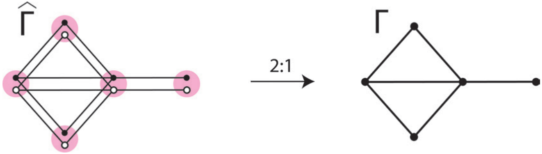

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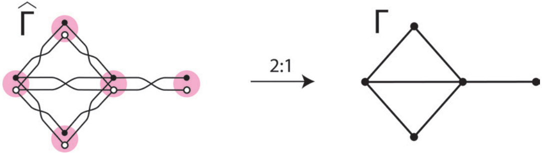
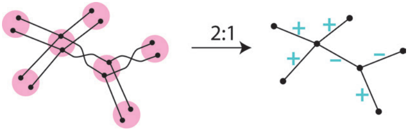


Double Covers of Trees and Cycles

Every double cover of a tree (or forest) is simply two copies of the original tree (or forest).



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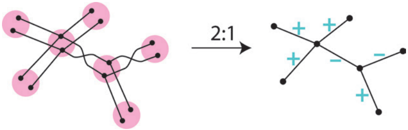

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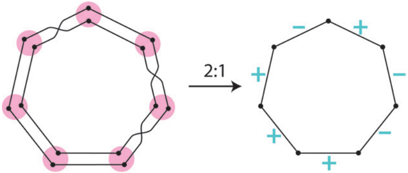

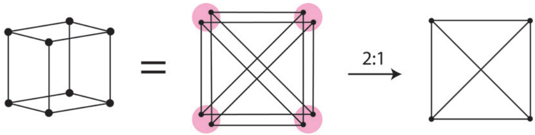

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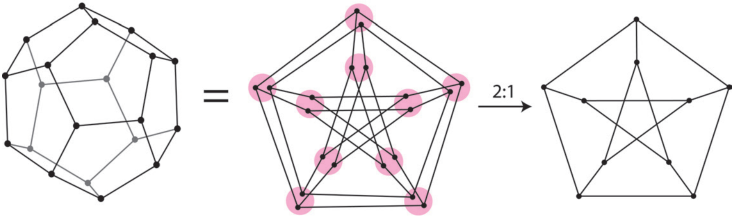

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Double Covers of Graphs

I have used double covers to

- construct objects,
- prove nonexistence of objects, and
- classify objects up to isomorphism.

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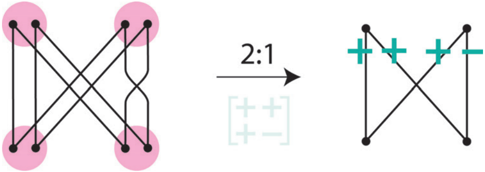
A double cover of $K_{n,n}$ is bipartite of degree n with $4n$ vertices. Each of the n pairs of vertices in one part, is joined to each of the n pairs of vertices in the other part, by a matching.



The covering graph is antipodal distance regular of diameter 3, iff it is a Hadamard graph, corresponding to a Hadamard matrix of order n.

Double Covers of Complete Bipartite Graphs $K_{n,n}$

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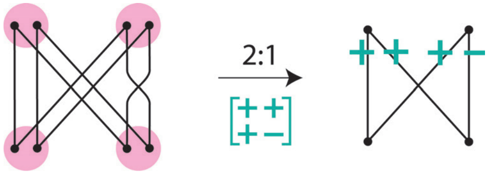

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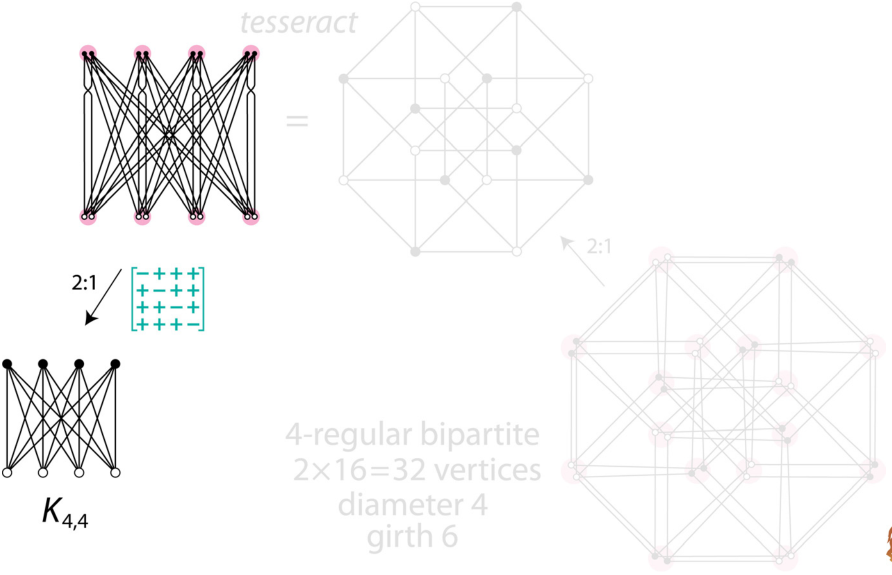
Sequential Double Covers of $K_{4,4}$



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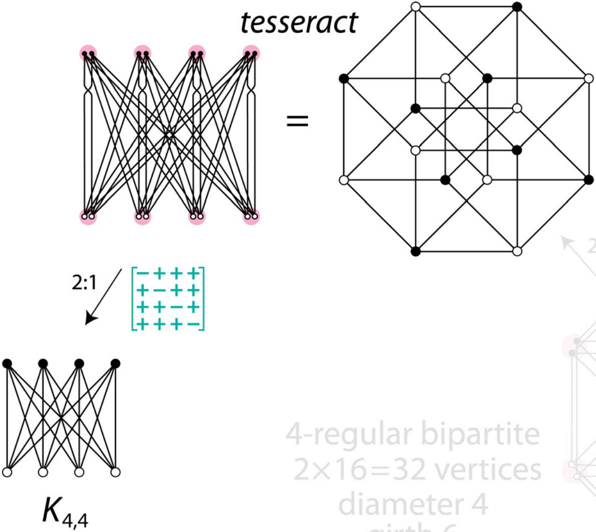

4-regular bipartite
2×16=32 vertices
diameter 4
girth 6

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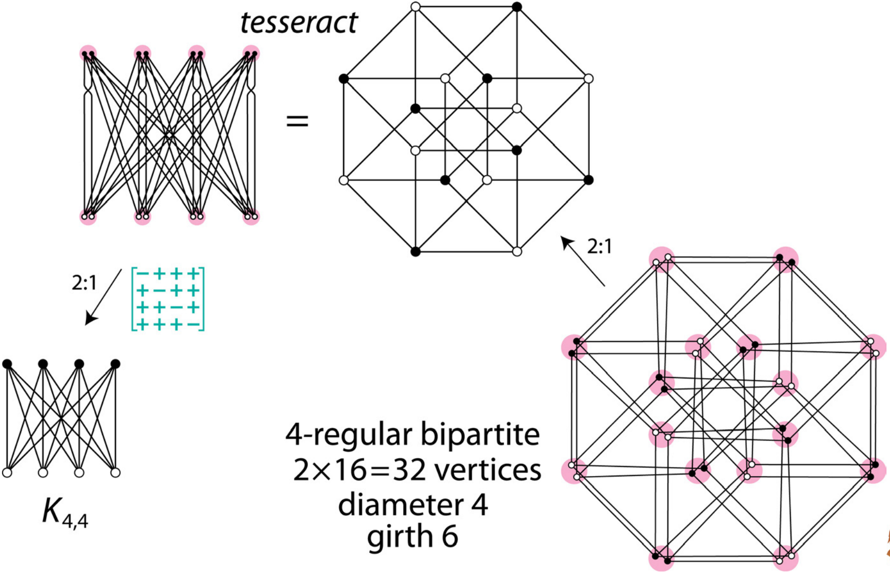

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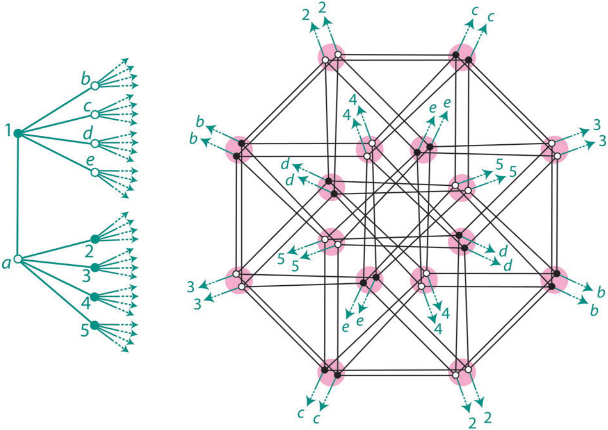
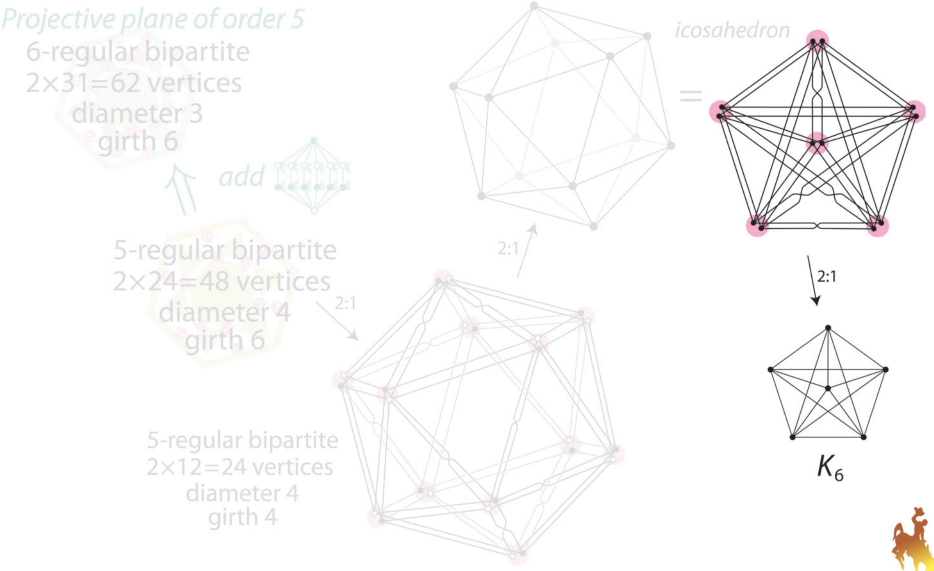


Projective plane of order 4

5-regular bipartite
2×21=42 vertices
diameter 3
girth 6

Sequential Double Covers of K_6



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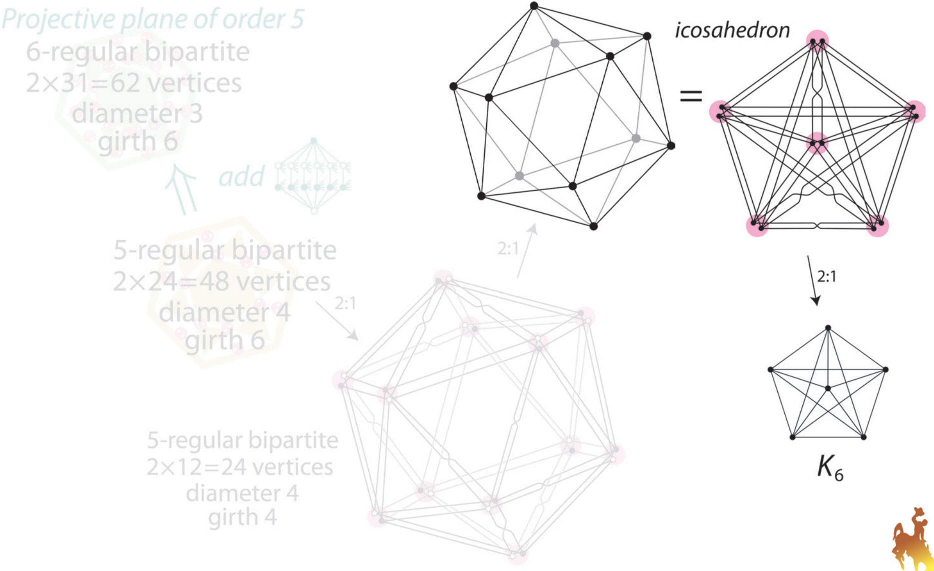

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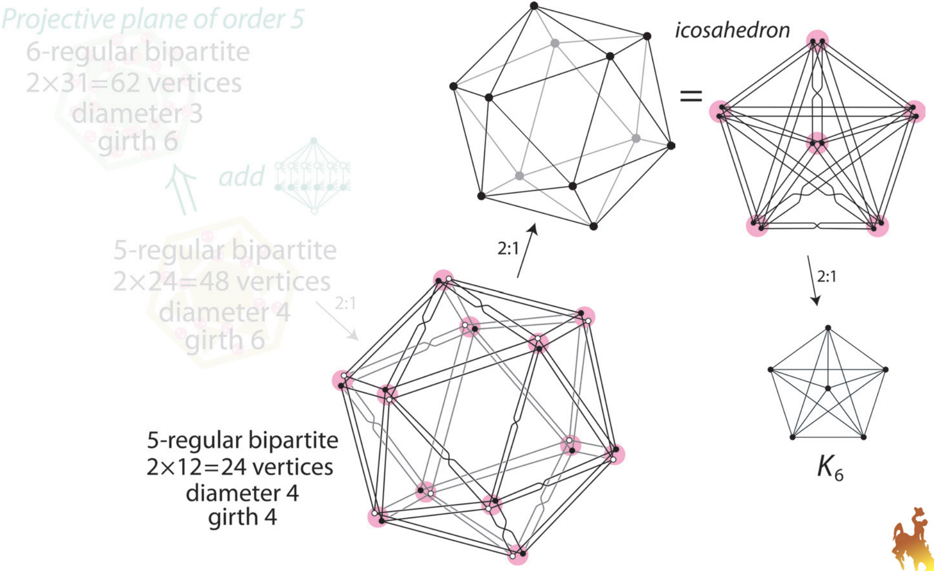


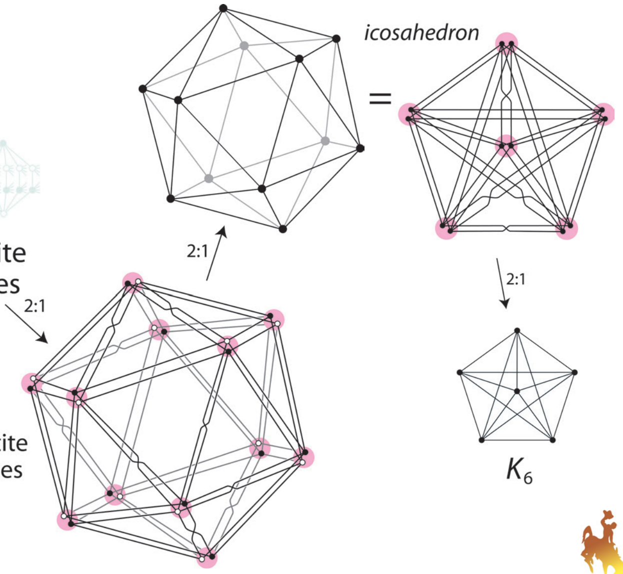
Sequential Double Covers of K_6

Projective plane of order 5
6-regular bipartite
2×31=62 vertices
diameter 3
girth 6

add

5-regular bipartite
2×24=48 vertices
diameter 4
girth 6

5-regular bipartite
2×12=24 vertices
diameter 4
girth 4



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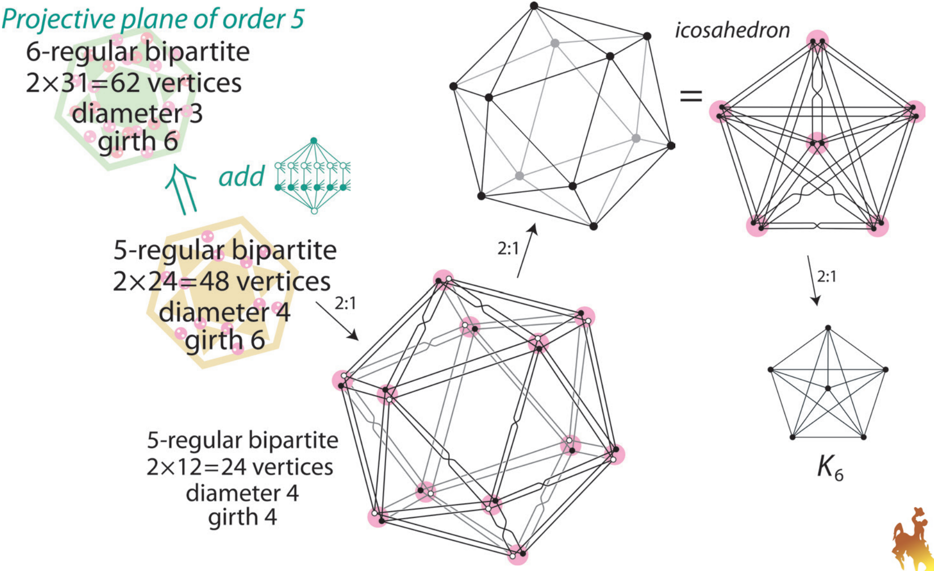
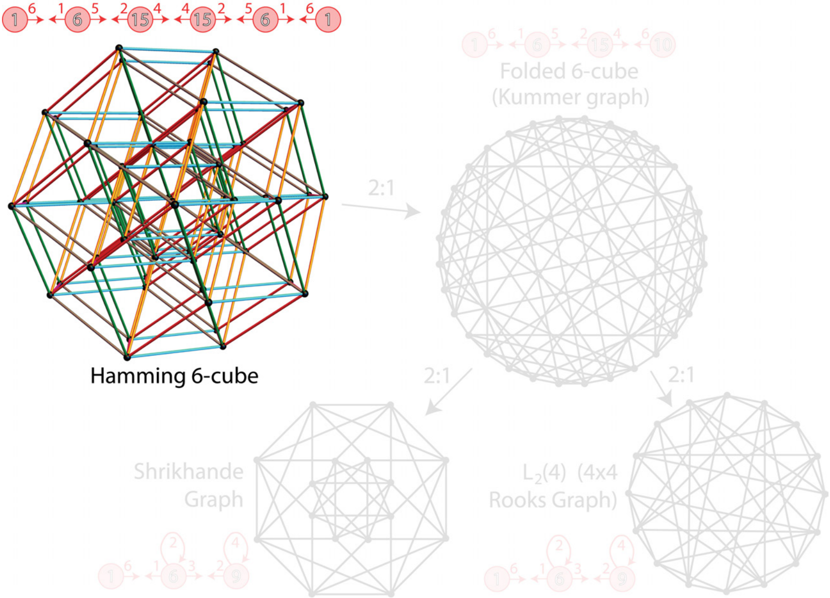


Sequential Double Covers Yielding SRG(16,6,2,2)'s



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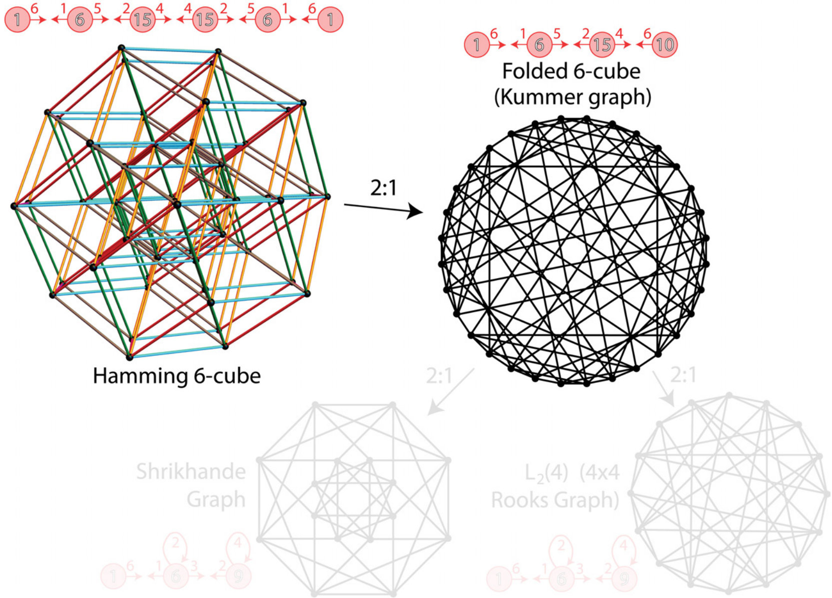

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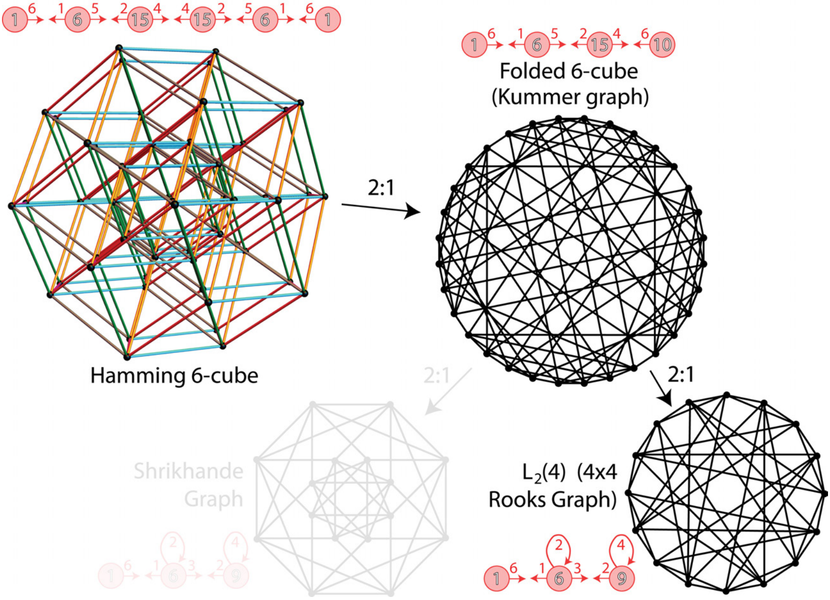

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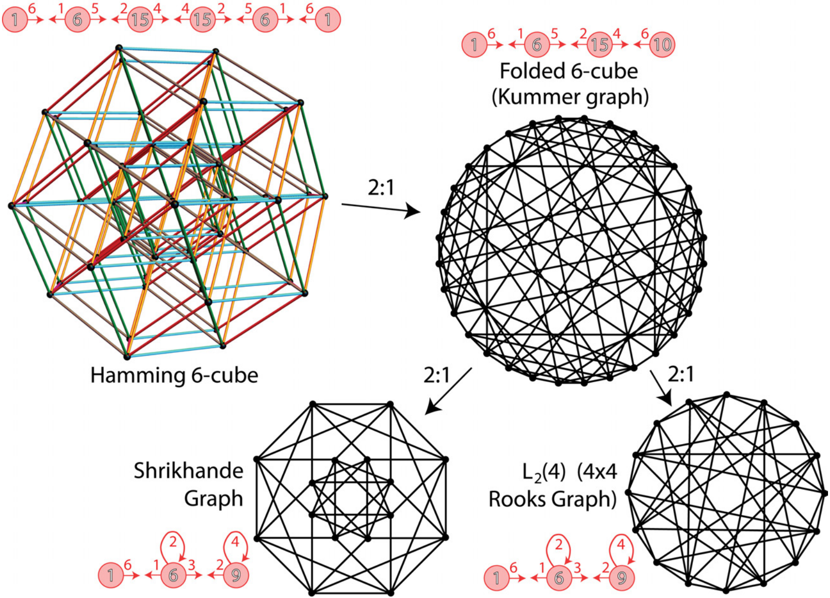

Constructing *new* projective planes

If Γ is chosen *carefully*, new projective planes arise from double covers $\widehat{\Gamma} \to \Gamma$ of girth 6.

The good news is, lifting Γ to $\widehat{\Gamma}$ is computationally *easy*.

But how do we obtain good candidates for Γ?

Idea: take Γ as a quotient of a *known plane*. The double cover $\widehat{\Gamma} \to \Gamma$ (of girth 6) is not necessarily unique.

Unfortunately no new orders of projective planes have been found this way.

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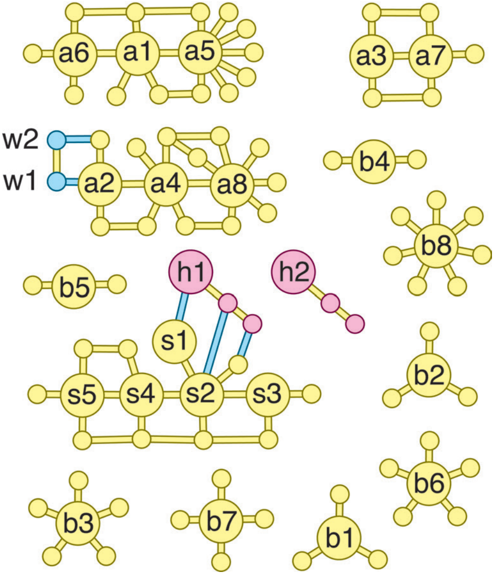
The known projective planes of order 25

99 planes of order 25 are known up to isomorphism/duality

(193 planes up to isomorphism)



Ovoids in finite orthogonal spaces

Let $V = F^n$ where $F = \mathbb{F}_q$ is a field of odd order q, and let $B : V \times V \to F$ be a nondegenerate symmetric bilinear form. The associated quadratic form is $Q(v) = \frac{1}{2}B(v, v)$.

Consider a collection of points (1-spaces) $\langle v_1 \rangle, \ldots, \langle v_N \rangle$ such that

$$B(v_i, v_j) = 0 \quad \text{iff } i = j.$$

We say $\mathcal{O} = \{\langle v_1 \rangle, \ldots, \langle v_N \rangle\}$ is an ovoid if $N = |\mathcal{O}|$ attains a certain upper bound (from easy counting arguments). Problem: existence? nonexistence? classify up to isomorphism?

Define a double cover $\widehat{K}_N \to K_N$ where

or

according as $B(v_i, v_j)$ is a square or a nonsquare. This is invariant under Aut($\mathcal{O}$), the stabilizer of in the orthogonal group. This is extremely effective for classification; and yields nonexistence for $n = 9, 11, 13, 15, \ldots$.

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A projective plane of order n can be represented by its bipartite *nonincidence* graph Γ. Here Γ is n^2-regular, bipartite with $2N$ vertices, $N = n^2 + n + 1$.

We define a double cover $\widehat{\Gamma} \to \Gamma$ using signs of perspectivities arising from antiflags. Using nauty/traces to test for isomorphisms and automorphisms is *much* more effective with $\widehat{\Gamma}$ than with the plane directly.

Similar techniques apply to other structures (latin squares and nets, loops, etc.).

Classifying known projective planes of small order

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Double covers of dual polar graphs

Only two infinite families are known of imprimitive Q-polynomial (but not P-polynomial) association schemes with an unbounded number of classes:

- bipartite doubles of Hermitian dual polar graphs (Bannai and Ito, 1984); and
- nontrivial double covers of symplectic dual polar graphs arising from the Maslov index (M. and Williford, 2015).

The Maslov Index

Let $V = F^{2n}$ where $F = \mathbb{F}_q$ is a field of order $q \equiv 1 \mod 4$. Fix a symplectic (i.e. nondegenerate alternating bilinear) form

$$B : V \times V \to F.$$

Let $\mathfrak{L}$ be the set of n-dimensional totally isotropic subspaces ('Lagrangians') of V. The symplectic dual polar graph $\Lambda_n(F)$ has vertex set $\mathfrak{L}$ and distance relation

$$d(X, Y) = \dim(X/X \cap Y) = \dim(Y/X \cap Y) \quad \text{for } X, Y \in \mathfrak{L}.$$

This graph is distance regular of diameter n.

For $X, Y, Z \in \mathfrak{L}$, one defines the Maslov index

$$\tau(X, Y, Z) \in \mathbb{Z}/2\mathbb{Z} \quad (\pm 1 \text{ written additively}).$$

This splits each of the distance relations, obtaining a $(2n+1)$-class Q-polynomial association scheme (M. and Williford, 2015).

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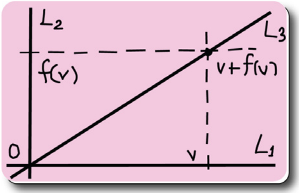
The Maslov Index

Given a triple of mutually transverse Lagrangians L_1, L_2, L_3 (i.e. $L_i \cap L_j = 0$ for $i \neq j$) there is a unique $f : L_1 \xrightarrow{\cong} L_2$ such that

$$L_3 = \{v + f(v) : v \in L_1\}.$$



Since L_3 is totally isotropic,

$$B(u+f(u), v+f(v)) = 0 \quad \text{for all } u, v \in L_1,$$

i.e. $B(u, f(v)) = B(v, f(u))$, $u, v \in L_1$. Since $\text{char}(F) \neq 2$ this symmetric bilinear for on L_1 gives a unique quadratic form

$$Q(u) = \tfrac{1}{2}B(u, f(u)), \quad u \in L_1.$$

Define the Maslov index $\tau(L_1, L_2, L_3)$ to be the class of Q in Witt(F), the Witt ring of F.

The Maslov Index

Given a triple of mutually transverse Lagrangians L_1, L_2, L_3 (i.e. $L_i \cap L_j = 0$ for $i \neq j$) there is a unique $f : L_1 \xrightarrow{\cong} L_2$ such that

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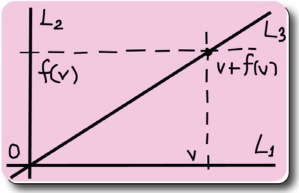

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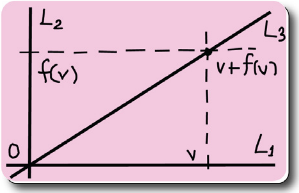

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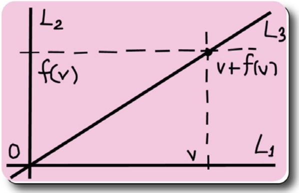

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Surprisingly, the Maslov index $\tau(L_1, L_2, L_3)$ can be defined for *all* Lagrangian triples (not just mutually transverse Lagrangians) satisfying the cocycle condition

$$\tau(L_1, L_2, L_3) - \tau(L_1, L_2, L_4) + \tau(L_1, L_3, L_4) - \tau(L_2, L_3, L_4) = 0$$

as well as G-invariance, $G = Sp_{2n}(F)$:

$$\tau(gL_1, gL_2, gL_3) = \tau(L_1, L_2, L_3), \quad L_i \in \mathfrak{L}, \; g \in G.$$

Two triples $(L_1, L_2, L_3), (L_1', L_2', L_3')$ are in the same G-orbit iff they are in the same $GL_{2n}(F)$-orbit $(\dim(L_i \cap L_j) = \dim(L_i' \cap L_j'))$ *and* $\tau(L_1, L_2, L_3) = \tau(L_1', L_2', L_3')$.

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For $L, L' \in \mathfrak{L}$, M. and Williford (2015) define $\sigma(L, L') = \sigma(L', L) \in \{\pm 1\}$ such that

$$\sigma(L, L', L'') := \sigma(L, L')\sigma(L', L'')\sigma(L'', L) = (-1)^{\tau(L,L',L'')}.$$

Thus σ defines a graph, and τ the associated two-graph, on $\mathfrak{L}$.

Vertices of the double cover $\widetilde{\Lambda}_n(F)$ are pairs $(L, \varepsilon) \in \mathfrak{L} \times \{\pm 1\}$ with adjacency

$$(L, \varepsilon) \sim (L', \varepsilon') \leftrightarrow \begin{cases} L \sim L', \text{ i.e. } \dim L/L \cap L' = \dim L'/L \cap L' = 1 \\ \quad \text{and} \\ \varepsilon\varepsilon' = \sigma(L, L'). \end{cases}$$

This is but one of the relations in a $(2n+1)$-class association scheme naturally defined on $\mathfrak{L} \times \{\pm 1\}$, invariant under $2 \times G = 2 \times Sp_{2n}(F)$.

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The case of $\Lambda_n(F)$, $F = \mathbb{F}_q$, $q \equiv 3 \mod 4$

If $q \equiv 3 \mod 4$ then

$$\tau(L', L'', L) = \tau(L, L', L'') = -\tau(L', L, L'')$$

so τ defines a skew two-graph (oriented two-graph) on $\mathfrak{L}$.

This gives a commutative (but non-symmetric) $(2n+1)$-class association scheme with vertices $(L, \varepsilon) \in \mathfrak{L} \times \{\pm 1\}$ as before.

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The case $n = 1$

For q odd, $\Lambda_1(\mathbb{F}_q) = K_{q+1}$, the complete graph on $q+1$ vertices.

For $q \equiv 1 \mod 4$, τ defines the Taylor graph $\widetilde{\Lambda}_1(\mathbb{F}_q) \to K_{q+1}$ corresponding to the Paley graph on $q+1$ vertices.

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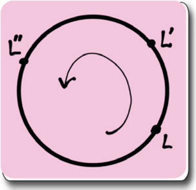
The real Lagrangian manifold $\Lambda_n(\mathbb{R})$

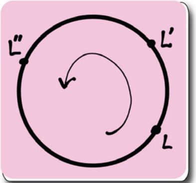


$\Lambda_1(\mathbb{R}) \simeq S^1$

$\sigma(L, L', L'') = (-1)^{\tau(L,L',L'')} = \pm 1$ according as the triple is oriented counter-clockwise or clockwise

$\pi_1(\Lambda_n(\mathbb{R})) \cong \mathbb{Z}$

From the 1-dimensional perspective, $\Lambda_n(\mathbb{R})$ looks like a circle.



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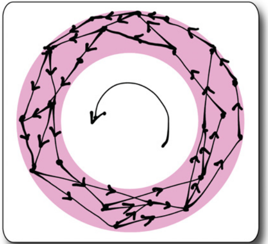

Ramanujan covering graphs

A currently popular approach to constructing new Ramanujan graphs is via sequential double covers of smaller (possibly trivial) Ramanujan graphs (Marcus, Spielman and Srivastiva, 2014; Bilu and Linial, 2004.)

A Few References

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