Colouring the Plane

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Chromatic Number of the Euclidean Plane

Consider the Euclidean plane $\mathbb{R}^2$ to be a graph with adjacency defined by the distance-one relation

$$(x, y) \sim (x', y') \quad \text{iff} \quad (x' - x)^2 + (y' - y)^2 = 1.$$

The chromatic number $\chi(\mathbb{R}^2)$ is the minimum number of colours needed to colour the points of $\mathbb{R}^2$ such that no two points at distance one bear the same colour. Known: $\chi(\mathbb{R}^2) \in \{4, 5, 6, 7\}$

$\chi(\mathbb{R}^2) \geqslant 4$ as seen from the *Moser spindle*:

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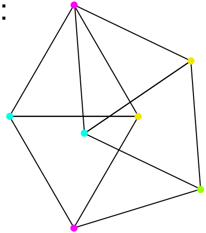

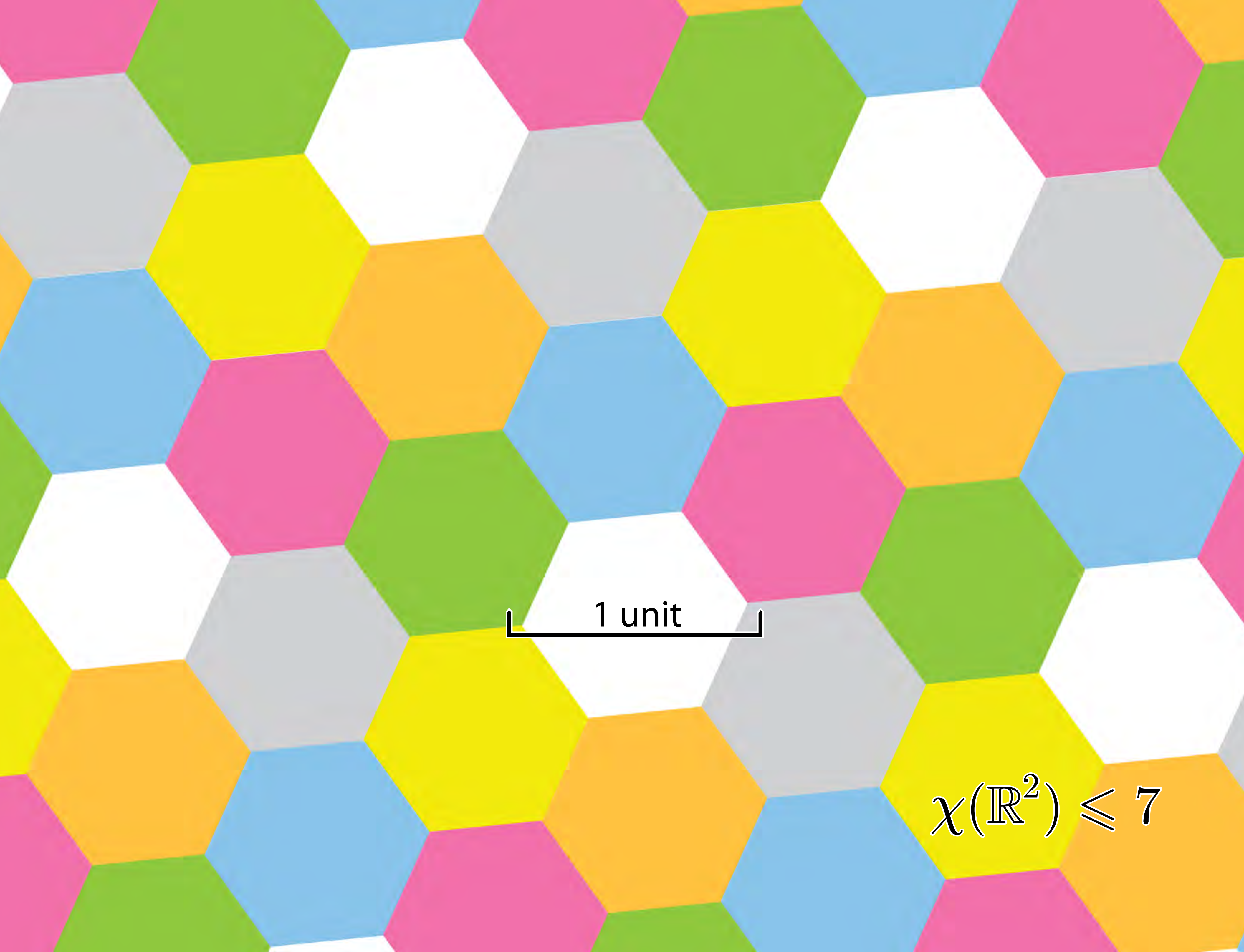
1 unit
$\chi(\mathbb{R}^2) \leqslant 7$

Chromatic Number of the Affine Plane K^2

Let K be an arbitrary field (or any commutative ring with 1).
Adjacency in K^2:

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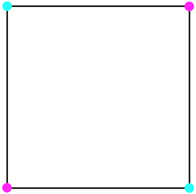

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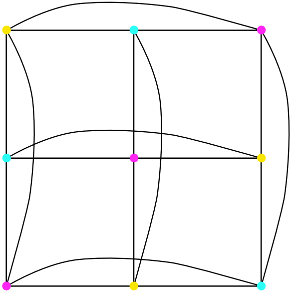

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Consider the subring

$$R = \left\{ \frac{a}{b} : a, b \in \mathbb{Z},\ b \text{ is a product of primes } \equiv 3 \bmod 4 \right\} \subset \mathbb{Q}.$$

The connected component of $(0,0)$ in $\mathbb{Q}^2$ is R^2. Note that $R/2R \cong \mathbb{F}_2$. There is a graph homomorphism $R^2 \to \mathbb{F}_2^2$:

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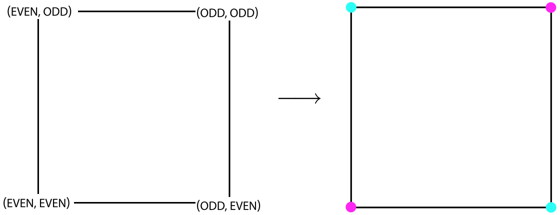

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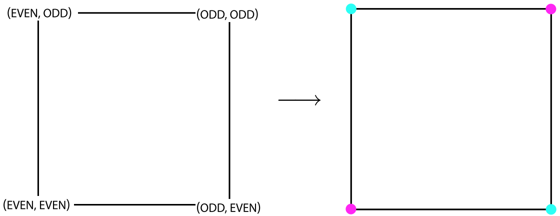

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$\chi(\mathbb{R}^2) = \chi(K^2)$ for a small subfield $K \subseteq \mathbb{R}$

By a theorem of de Bruijn and Erdös, $\chi(\mathbb{R}^2) = \chi(\Gamma)$ for some finite subgraph $\Gamma \subset \mathbb{R}^2$.

Let $K \subset \mathbb{R}$ be the subfield generated by the coordinates of all vertices in K.

So $\chi(\mathbb{R}^2) = \chi(K^2)$ where the subfield $K \subset \mathbb{R}$ is finitely generated over $\mathbb{Q}$. In particular K is countable.

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$\chi(\mathbb{R}^2) = \chi(K^2)$ for a finite extension $K \supset \mathbb{Q}$

Theorem (M.)

There exists a number field K embeddable in $\mathbb{R}$ and subfields

$$K = K_n \supset K_{n-1} \supset \cdots \supset K_1 \supset K_0 \supseteq \mathbb{Q}$$

such that

(a) $\chi(\mathbb{R}^2) = \chi(K^2)$

(b) $[K_i : K_{i-1}] = 2$ *for* $i = 1, 2, \ldots, n$

(c) *the extension* $K_0 \supseteq \mathbb{Q}$ *is finite of odd degree* $[K_0 : \mathbb{Q}]$

(d) $\chi(K_0^2) = \chi(\mathbb{Q}^2) = 2$

Note that points of K^2 are straightedge-and-compass constructible from points of K_0^2.

Does it make sense to colour by induction on n? Each 'new' point in K_i^2 (i.e. not in K_{i-1}^2) has at most two neighbours in K_{i-1}^2.

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Quadratic extensions

We are especially interested in how much the chromatic number can grow for quadratic extensions.

Note:

K^2 contains a 3-cycle iff $K \supseteq \mathbb{Q}(\sqrt{3}\,)$.

$\chi(\mathbb{Q}(\sqrt{3}\,)^2) = 3$.

K^2 contains a Moser spindle iff $K \supseteq \mathbb{Q}(\sqrt{3}, \sqrt{11}\,)$.

$\chi(\mathbb{Q}(\sqrt{3}, \sqrt{11}\,)^2) \in \{4, 5, 6, 7\}$.

What can we say about $\chi(K^2)$ when $K = \mathbb{Q}(\sqrt{d}\,)$, $d \geqslant 2$ a squarefree integer?

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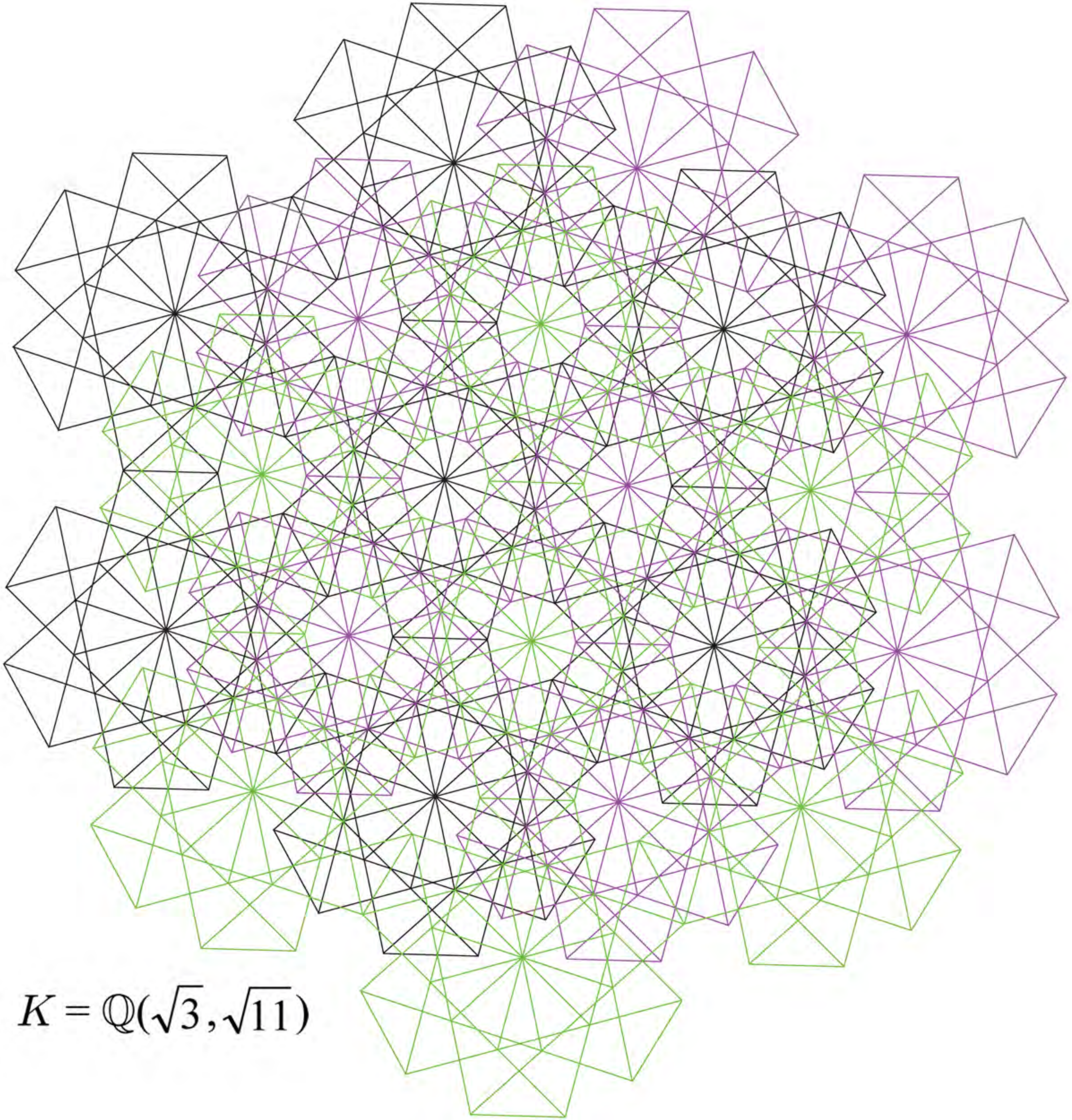

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$\chi(\mathbb{F}_q^2)$ for finite fields $\mathbb{F}_q$

$$\mathbb{F}_q^2 \text{ is regular of degree } \begin{cases} q, & \text{for } q \text{ even;} \\ q-1, & \text{if } q \equiv 1 \mod 4; \\ q+1, & \text{if } q \equiv 3 \mod 4. \end{cases}$$

If q is even, then $\mathbb{F}_q^2$ is a disjoint union of $q/2$ complete bipartite graphs $K_{q,q}$, and $\chi(\mathbb{F}_q^2) = 2$.

If q is odd, then $\chi(\mathbb{F}_q^2) \geqslant 3$.

If $q \equiv \pm 1 \mod 12$, then $\chi(\mathbb{F}_q^2) \geqslant 4$.

q	3	5	7	9	11	13	17
$\chi(\mathbb{F}_q^2)$	3	3	4	3	5	5 or 6	5, 6 or 7

Similar computations have been done by Sebastian Cioabă and Jason Williford.

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$\chi(K^2)$ for quadratic extensions $K \supset \mathbb{Q}$

Theorem (M.)

Let $K = \mathbb{Q}(\sqrt{d}\,)$, $d \geqslant 2$ a squarefree integer.
If $d \not\equiv 47, 59$ or $83 \mod 84$, then $\chi(K^2) \leqslant 4$.

What about $\chi(\mathbb{Q}(\sqrt{47}\,)^2)$?

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Apparently open question: Is $\chi(\mathbb{C}^2) < \infty$?

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Let K be a number field. Then

(a) *K^2 is disconnected iff $i \in K$ where $i = \sqrt{-1}$.*

(b) *If $i \notin K$ then $\chi(K^2) < \infty$.*

Is $\chi(K^2) < \infty$ where $K = \mathbb{Q}(i)$, $i = \sqrt{-1}$?

Other subfields $K \subseteq \mathbb{C}$

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