Semifinite Generalized Quadrangles

G. Eric Moorhouse

Department of Mathematics
University of Wyoming

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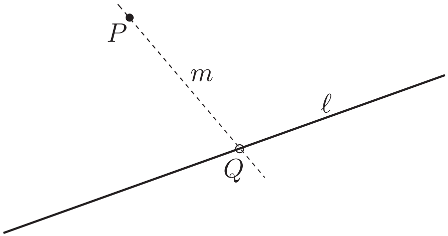
Generalized Quadrangles

A *generalized quadrangle* (GQ) is a point-line incidence structure in which every non-incident point-line pair (P, ℓ) has exactly one line through P meeting ℓ:



We assume every point is on more than two lines; and every line has more than 2 points.

Semifinite Generalized Quadrangles

We say the GQ is *semifinite* if it has infinitely many points and lines, but the number of points on each line (always the same number) is $k<\infty$. (*Open question: Can this happen?*)

There is no semifinite GQ with line size $k=3$ (Cameron, 1981 . . . one paragraph).

There is no semifinite GQ with line size $k=4$ (Brouwer, 1991 . . . three pages).

There is no semifinite GQ with line size $k=5$ (Cherlin, 2005 . . . seven pages of model theory).

Nothing is known for line size $k\geqslant 6$. Experts differ on whether semifinite GQ's may exist at all.

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First-Order Axioms for Generalized Quadrangles

$P(x), L(x)$: unary predicates
$I(x,y)$: binary predicate

1. $(\forall x)(P(x) \leftrightarrow \neg L(x))$
2. $(\forall x)(\forall y)\big(I(x,y) \to (P(x) \wedge L(y))\big)$
3. $(\forall p)(\forall \ell)\big((P(p) \wedge L(\ell)) \to (\exists q)(\exists m)(I(p,m) \wedge I(q,m) \wedge I(q,\ell)\big)$
4. $(\forall p_1)(\forall p_2)(\forall \ell_1)(\forall \ell_2)$
   $\big((I(p_1,\ell_1) \wedge I(p_1,\ell_2) \wedge I(p_2,\ell_1) \wedge I(p_2,\ell_2)) \to (p_1 = p_2 \wedge \ell_1 = \ell_2)\big)$
5. $(\forall p_1)(\forall p_2)(\forall p_3)(\forall \ell_1)(\forall \ell_2)(\forall \ell_3)\ \big((I(p_1,\ell_1) \wedge I(p_1,\ell_2) \wedge I(p_2,\ell_2) \wedge I(p_2,\ell_3) \wedge I(p_3,\ell_3) \wedge I(p_3,\ell_1)) \to (p_1 = p_2 \wedge p_1 = p_3 \wedge p_2 = p_3 \wedge \ell_1 = \ell_2 \wedge \ell_1 = \ell_3 \wedge \ell_2 = \ell_3)\big)$
6. $(\exists p_1)(\exists p_2)(\exists p_3)(\exists \ell)$
   $\big(I(p_1,\ell) \wedge I(p_2,\ell) \wedge I(p_3,\ell) \wedge p_1 \neq p_2 \wedge p_1 \neq p_3 \wedge p_2 \neq p_3\big)$
7. similarly, each point is on at least 3 lines

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To say that the line size is 3, add an axiom

- $(\forall p_1)(\forall p_2)(\forall p_3)(\forall p_4)(\forall \ell)$ $\big((I(p_1,\ell) \wedge I(p_2,\ell) \wedge I(p_3,\ell) \wedge I(p_4,\ell))$ $\to (p_1{=}p_2 \vee p_1{=}p_3 \vee p_1{=}p_4 \vee p_2{=}p_3 \vee p_2{=}p_4 \vee p_3{=}p_4\big)$

To say that there are infinitely many points and lines, add an infinite list of axioms

- $(\exists x_1)(\exists x_2)(x_1{\neq}x_2)$
- $(\exists x_1)(\exists x_2)(\exists x_3)(x_1{\neq}x_2 \ \vee \ x_1{\neq}x_3 \ \vee \ x_2{\neq}x_3)$
- etc.

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  $\rightarrow (p_1{=}p_2 \vee p_1{=}p_3 \vee p_1{=}p_4 \vee p_2{=}p_3 \vee p_2{=}p_4 \vee p_3{=}p_4)$

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- etc.

Model Theory as a Convenient Crutch

Proofs by model-theoretic methods are sometimes shorter or more natural than alternative proofs obtained by other means.

Concern may be expressed over the liberal use of the axiom of choice (AC) in model theory. But often, proofs obtained by these methods can be rewritten so as to obtain more ‘constructive’ proofs not requiring AC.

The model-theoretic language often serves as a convenience rather than as a necessity. Its use is similar to proofs in discrete mathematics that appeal to $\mathbb{R}$ or to $\mathbb{C}$, where typically a finite extension of $\mathbb{Q}$ suffices.

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Order Indiscernibles

Let T be a theory (say, a set of axioms) having a model $\mathcal{M}$ with underlying set M. (We write $\mathcal{M} \models T$.) An indexed family of distinct elements $m_j \in M$ (for $j \in \mathbb{Q}$) consists of *order indiscernibles* if for every propositional formula $\phi(x_1, \ldots, x_r)$, and every pair of increasing sequences

$$j_1 < j_2 < \cdots < j_r, \quad k_1 < k_2 < \cdots < k_r \quad \text{in } \mathbb{Q},$$

we have

$$\phi(m_{j_1}, m_{j_2}, \ldots, m_{j_r}) \text{ holds in } \mathcal{M} \text{ iff } \phi(m_{k_1}, m_{k_2}, \ldots, m_{k_r}) \text{ holds in } \mathcal{M},$$

i.e. $\mathcal{M} \models \big(\phi(m_{j_1}, m_{j_2}, \ldots, m_{j_r}) \leftrightarrow \phi(m_{k_1}, m_{k_2}, \ldots, m_{k_r})\big)$.

For example, in $\mathbb{C}$, any countably infinite algebraically independent subset consists of order indiscernibles (but in this case an arbitrary ordering can be used).

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Another Example of Order Indiscernibles

A more enlightening example is the theory of *dense linear orders without endpoints*. This has a single binary predicate $R(x,y)$ written as $x{<}y$, with axioms

1. $(\forall x)(\forall y)(\forall z)\big((x{<}y \ \wedge \ y{<}z) \to x{<}z\big)$
2. $(\forall x)(\forall y)(x{=}y \ \vee \ x{<}y \ \vee \ y{<}x)$
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For examples, $(\mathbb{R},<)$ and $(\mathbb{Q},<)$ are models. In both cases, $\mathbb{Q}$ is a set of order indiscernibles.

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Order Indiscernibles and Group Actions

Let $\mathrm{Aut}(\mathbb{Q},<)$ be the group of all order-preserving permutations (a *highly homogeneous permutation group*—transitive on subsets of size n for each $n \geqslant 1$).

This group accounts for the fact that the entire set $\mathbb{Q}$ consists of order indiscernibles in $(\mathbb{Q},<)$.

More generally: Given an indexed family of distinct elements $S = \{m_j : j \in \mathbb{Q}\} \subseteq \mathcal{M}$, the group $\mathrm{Aut}(\mathbb{Q},<)$ permutes S naturally (acting on subscripts). If every $g \in \mathrm{Aut}(\mathbb{Q},<)$ extends to an automorphism of $\mathcal{M}$, then S consists of order indiscernibles.

The converse fails; there exist sets of order indiscernibles in structures with no automorphisms.

However, it is usually possible to pretend that sets of order indiscernibles are subsets $S \subseteq \mathcal{M}$ with $(\mathrm{Aut}\,\mathcal{M})_S \cong \mathrm{Aut}(\mathbb{Q},<)$ acting naturally on subscripts.

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Existence of Order Indiscernibles

Theorem

If a theory T has some infinite model, then T has a model containing a set of order indiscernibles.

For example, if there exists a semifinite GQ, then there exists a semifinite GQ containing a set of order indiscernibles. This set must consist of

(i) a *partial spread* (i.e. set of mutually disjoint lines), or
(ii) a set of lines through a single point, or
(iii) a *cap* (i.e. a set of points, no two on the same line).

We may assume that case (i) occurs: a *partial spread of order indiscernible lines*.

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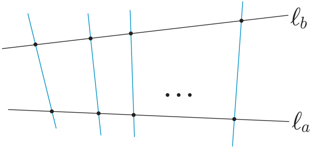
Partial Spread of Order Indiscernible Lines

Henceforth let $\mathcal{Q}$ be a semifinite GQ with k per line, and let $\{\ell_a : a \in \mathbb{Q}\}$ be a partial spread consisting of order indiscernible lines. Every pair of lines ℓ_a, ℓ_b ($a<b$) has exactly k transversals.



But:

Lemma

No three lines of our partial spread have a common transversal.

Proof: Suppose ℓ_a, ℓ_b, ℓ_c have a common transversal where $a<b<c$. By indiscernibility, ℓ_a, ℓ_b, ℓ_d have a common transversal whenever $a<b<d$. However, there are only k common transversals to ℓ_a and ℓ_b, each with only k points, a contradiction. ☐

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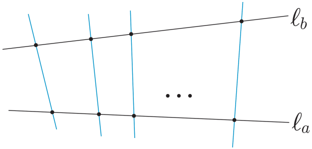

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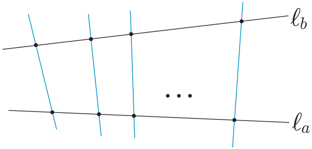

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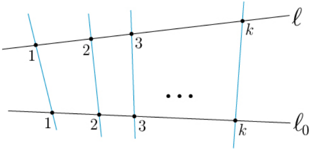
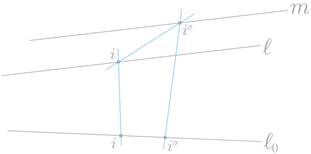
Partial Spread of Order Indiscernible Lines

Label the points of ℓ_0 as $1, 2, \ldots, k$. Using transversals, this induces a labeling of the points of each line ℓ disjoint from ℓ_0:



Whenever ℓ_0, ℓ, m are mutually disjoint lines, transversals induce a permutation $\sigma = \sigma(\ell, m) \in S_k$:



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The number of fixed points of $\sigma(\ell, m) \in S_k$ equals the number of common transversals of ℓ_0, ℓ, m. $\square$

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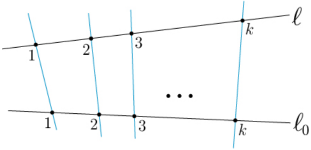

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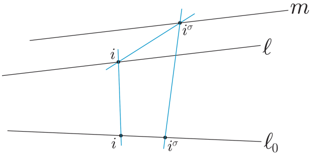

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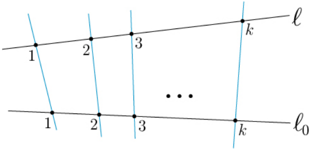

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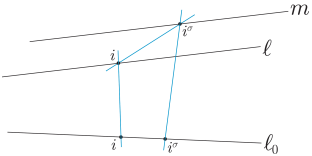

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Partial Spread of Order Indiscernible Lines

A similar argument shows

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Suppose ℓ_0, ℓ, m, n are mutually disjoint lines. Then the number of fixed points of $\sigma(\ell, m)\sigma(m, n)\sigma(n, \ell) \in S_k$ equals the number of common transversals of ℓ, m, n.

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Since ℓ_0, ℓ_1, ℓ_2 have no common transversals, the permutation $\sigma = \sigma(\ell_1, \ell_2)$ is fixed-point-free. Moreover by indiscernibility,

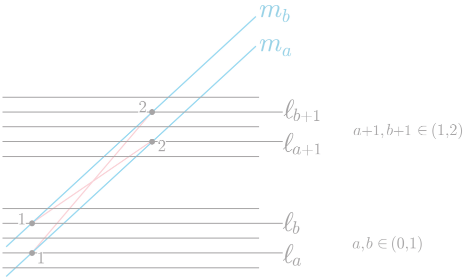
$$\sigma(\ell_a, \ell_b) = \sigma \quad \text{whenever } 0<a<b.$$

Lemma

Without loss of generality, $\sigma = (12)\widetilde{\sigma}$ where $\widetilde{\sigma}$ is a fixed-point-free permutation of $3, 4, \ldots, k$.

Proof: WLOG $1^\sigma = 2$.

Now $\{m_a : a \in \mathbb{Q} \cap (0,1)\}$ is a partial spread of order indiscernible lines with $\sigma(m_a, m_b) = (12)(\cdots)$ whenever $0<a<b<1$. □



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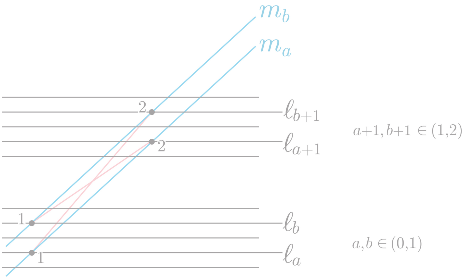

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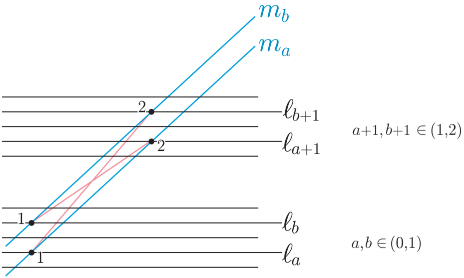

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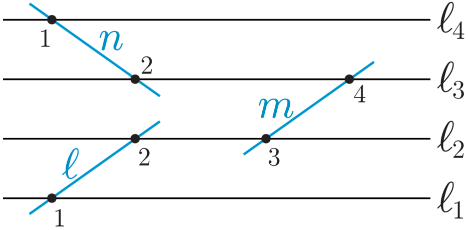

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So $\sigma(\ell, m)\sigma(m, n)\sigma(n, \ell) = (123)$, which has one fixed point. Thus ℓ, m, n have a common transversal. This cannot happen, so

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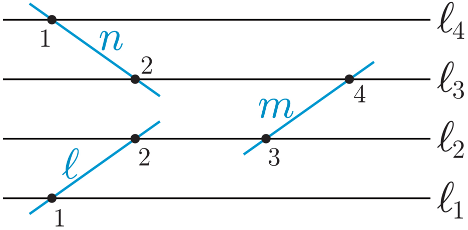

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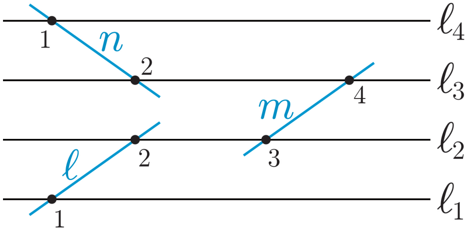

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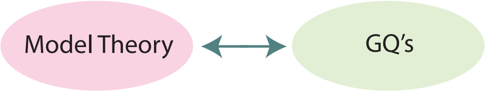
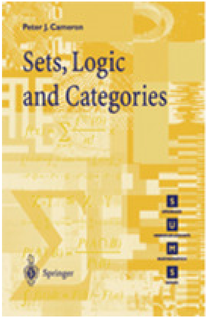
GQ’s and Model Theory

Cherlin applied techniques of model theory to the study of GQ’s.



However, the study of GQ’s is also applied to model theory. GQ’s arise in the investigation of the Cherlin-Zil’ber Conjecture, one of the leading open problems in model theory.

General References



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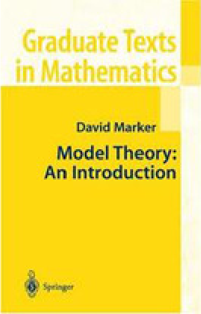

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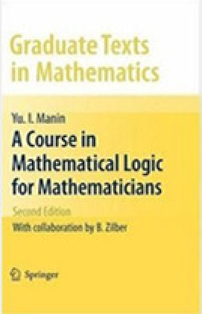

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